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Preface

Harmonic maps between Riemannian manifolds are canonical objects from the points of view of topology and calculus of variations. These maps provide a rich display of both differential geometric and analytic phenomena. Much of the study of these maps serves as a model for many other challenging problems in geometric analysis and has been the source of inspiration and undiminishing fascination.

The study of harmonic maps in one dimension is equivalent to the study of the shortest paths—geodesics in Riemannian manifolds. The classical Morse theory (see Milnor [143]) and the Lusternik-Schnirelmann theory [130] were created from the study of such objects.

Harmonic maps with two-dimensional domains present special features that are crucial for applications to minimal surfaces (i.e., conformal harmonic maps) and to the deformation theory of Riemann surfaces—Teichmüller theory (see Wolf [213]). We refer to books by Jost [103] and Helein [93] and surveys by Schoen [167, 168] for detailed and systematic presentations of the various aspects of the theory. Many of the geometric and analytic methods used in the study of two-dimensional harmonic mapping problems can be adapted and generalized to the study of other geometrical objects such as constant mean curvature surfaces, Willmore surfaces, and pseudo-holomorphic curves. This last topic has played a fundamental role in the study of four-dimensional topology as well as symplectic and Kähler manifolds; see for example Gromov [73] and McDuff and Salamon [144].

We would also point out that the study of harmonic diffeomorphisms between two-dimensional domains (see Jost [103], Hélein [89], Wan [204], Tam-Wan [199], Han-Tam-Treibergs-Wan [78], and Li-Tam-Wang [119]) is closely related to the study of another important classical geometry problem: isometric embedding. Indeed, Lewy [129] reduced the study of Monge-Ampère equations for isometric embeddings to the study of the Darboux system which describes a special class of generalized harmonic maps that led him to develop solutions of the Weyl isometric embedding and Minkowski problems in two-dimensions for analytic metrics. Heinz [87, 88] further generalized Lewy's approach, and the work of F. Labourie [113] can be viewed as an interpretation in the language of Gromov, see for example Lin [121].

The study of harmonic maps from a compact Riemannian manifold M into another compact Riemannian manifold N in higher dimensions probably began with the ground-breaking work of Eells and Sampson [49]. They proved, in particular, that any homotopy class of maps from M into N contains a smooth harmonic map whenever the target manifold N is nonpositively curved. The result has shown to be extremely useful for establishing certain rigidity and vanishing theorems which can be seen in Siu [190], Corlette [37], Jost and Yau [106, 107], and Gromov and Schoen [74]. There have been very important studies on harmonic maps from suitable metric spaces into Alexandrov spaces of nonpositively curvature by Korevaar and Schoen [111, 112], and Jost [104, 105]. There have been important works on both harmonic maps and their heat flows on complete, noncompact Riemannian manifolds into compact Riemannian manifolds with nonpositive curvature by Li and Tam [116, 117, 118].

The theory of harmonic maps is remarkably rich. It took "A report on harmonic maps" [50] in 1978 and "Another report on harmonic maps" [51] in 1988 by Eells and Lemaire to give a brief survey on the subject. These reports contain nearly one thousand relevant references. Since then there have been many more developments, especially during the past two decades. In addition to books by Jost [102, 103] and Helein [93] that we mentioned earlier, there are books by Giaquinta, Modica, and Souček [70, 71], Schoen and Yau [176], lecture notes by Struwe [196] and Simon [187, 188], and surveys by Schoen [167, 168, 169], Hardt [80], Brezis [16, 17] and Helein [94]. Therefore, it is almost impossible to write a book on harmonic maps that will be a comprehensive and complete account of the entire theory.

Our goal in this book is to present a significant portion of the analytic aspects of the theory of harmonic maps and the associated heat flows. These ideas and techniques are central to the development of many other related studies on the general Gauge theory, the theory of liquid crystals, and the theory of Ginzburg-Landau equations. We shall not discuss these theories or their applications. We shall also omit entirely a discussion of the geometric aspects of harmonic maps and various beautiful and important applications. Interested readers may find some of the references mentioned herein to be quite informative on these topics.

Organization of the book

Our book is organized as follows. In Chapter 1, after a brief introduction, we derive the Euler-Lagrange equations for harmonic maps from both intrinsic and extrinsic views. We then derive the Bochner identity and the second variational formula for harmonic maps. These calculations will be very useful later on, especially in the study of stable harmonic maps.

Chapter 2 is devoted to the regularity theory of energy minimizing harmonic maps between two Riemannian manifolds. It includes (i) Morrey's classical regularity theorem for energy minimizing maps defined on two-dimensional domains; (ii) partial regularity theorems for energy minimizing maps from a domain of dimension at least 3 due to Schoen and Uhlenbeck [171, 172] (see also Giaquinta and Giusti [66]); and (iii) the uniqueness of minimizing tangent maps at an isolated singularity due to L. Simon [181, 182, 183].

We discuss the partial regularity theory for weakly harmonic maps or stationary harmonic maps in Chapter 3. We begin with a classical result of Hildebrandt, Kaul and Widman [95] concerning weakly harmonic maps into a geodesically convex neighborhood of a point of the target manifold. We note that Giaquinta and Giusti [67], and Caffarelli [20] also established similar results via rather different arguments. We then present in Section 3.2 the beautiful theorem of Helein on the regularity of weakly harmonic maps on two-dimensional domains. In higher dimensions, under the assumption that these weakly harmonic maps are also stationary, Evans [45] and Bethuel [11] proved a partial regularity theorem that we describe in Section 3.3. We point out that Riviere [160] constructed examples of nowhere smooth weakly harmonic maps into spheres whenever the domain dimension is at least 3. In the final section of this Chapter, we present some optimal partial regularity theorems for stable-stationary harmonic maps. In particular, we will discuss the contributions by Schoen and Uhlenbeck [174], and recent works of Hong-Wang [98], and the authors [132].

The blow-up analysis for stationary harmonic maps is carried out in Chapter 4 where we study a sequence of weakly convergent stationary harmonic maps. We first establish the rectifiability of the energy concentration sets for such a sequence. Then we derive a necessary and sufficient condition for such a sequence to be compact in the Sobolev space $H^1(M, N)$ of maps. Consequently, we also obtain some necessary and sufficient conditions for uniform gradient estimates in terms of the energy of such maps. The final statement extends the earlier theorems of Eells and Sampson, Schoen and Uhlenbeck, Giaquinta and Giusti, and Hildebrandt, Kaul and Widman.

Chapter 5 is devoted to the theory of Eells and Sampson on the existence of global smooth heat flows of harmonic maps into compact Riemannian manifolds of nonpositive curvature. The results have been generalized by Mayer [142] to general nonpositively curved metric space.

Harmonic maps from two-dimensional domains have various special features. Due to conformal invariance of the Dirichlet integral, harmonic maps are critical for analysis. In Chapter 6, we carry out bubbling analysis that was initiated in the work of Sacks-Uhlenbeck [164] and was then generalized by M. Struwe [193] for heat flows in two dimensions. These works are discussed in Sections 6.1 and 6.2. In 6.3 we present an example of finite time blow-up by Chang-Ding-Ye [23]. In Sections 6.4 and 6.5 we describe optimal results by the authors for the bubble tree convergence for heat flow of harmonic maps at a finite time singularity and energy equality in the bubbling process. The final result improved earlier results of Jost [103], Parker [152], Parker-Wolfson [153], Qing [157], Ding-Tian [43] and others.

Chapter 7 is devoted to the theory developed by Chen and Struwe [33] that leads to the existence of partially smooth heat flows of harmonic maps. Under some additional assumptions, such weak flows will be smooth. Otherwise, there are counterexamples for global existence of smooth solutions as well as uniqueness of such flows.

The last two Chapters are devoted to a more detailed measure theoretic study on the Chen-Struwe's weak solutions. We begin with a necessary and sufficient condition for the Eells-Sampson uniform gradient estimate. Similar results are also true for the heat flow of Ginzburg-Landau type equations. In order to do that, we first need to characterize obstructions to the strong convergence of solutions (or approximate solutions) to the heat-flow of harmonic maps. Then we can apply Chen-Struwe [33] and Struwe's [195] monotonicity formula along with the Almgren-Federer stratification and dimension reduction principle to study the energy concentration phenomenon. The parabolic version of the rectifiability theorem of the energy concentration sets is then proved in Section 8.3. We then establish the main generalized version of Eells-Sampson theorems.

In the final Chapter, we establish a generalized varifold flow and show that the energy concentration sets evolve according to the motion by mean-curvature of Brakke [15]. We also establish parabolic versions of the energy quantization theorems. We believe similar results may be established for other geometric flow problems.

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Chapter 1

Introduction to harmonic maps

1.1 Dirichlet principle of harmonic maps

Harmonic maps are nonlinear extensions of harmonic functions. Just like harmonic functions, harmonic maps are critical points of a natural energy functional, called Dirichlet energy, of maps between two Riemannian manifolds.

Let (M, g) be a n-dimensional Riemannian manifold with or without boundary, endowed with a smooth Riemannian metric g. For any fixed point $p_0 \in M$, let $(x_1, \cdots, x_n)$ be a coordinate system near p_0 so that g can be represented by

$$g = \sum_{1 \le \alpha, \beta \le n} g_{\alpha\beta}\, dx_\alpha\, dx_\beta,$$

where $(g_{\alpha\beta})$ is a positive definitive symmetric $n \times n$ matrix. Let $(g^{\alpha\beta}) = (g_{\alpha\beta})^{-1}$ be the inverse matrix of $(g_{\alpha\beta})$ and $dv_g = \sqrt{g}\, dx = \sqrt{\det(g_{\alpha\beta})}\, dx$ be the volume element of (M, g). Let (N, h) be a l-dimensional compact Riemannian manifold without boundary which is endowed with a smooth Riemannian metric h.

Throughout this book we use the Einstein convention for summation. For any map $u \in C^2(M, N)$, we can define its Dirichlet energy as follows. For any fixed $p \in M$, there exist two normal coordinate charts $U_p \subset M$ of p and $V_q \subset N$ of $q = u(p)$ such that $u(U_p) \subset V_q$. The Dirichlet energy density function $e(u)$ is defined by

$$e(u)(x) \left(\equiv |\nabla u|_g^2\right) = \frac{1}{2} \sum_{\alpha,\beta} g^{\alpha\beta}(x) h_{ij}(u(x)) \frac{\partial u^i}{\partial x_\alpha} \frac{\partial u^j}{\partial x_\beta}, \tag{1.1}$$

where (x_α) and (u^i) are the coordinate systems on U_p and V_q respectively. The Dirichlet energy functional is defined by

$$E(u) = \int_M e(u)\, dv_g. \tag{1.2}$$

Definition 1.1.1 A map $u \in C^2(M, N)$ is a harmonic map, if it is a critical point of the Dirichlet energy functional E.

We first have

Proposition 1.1.2 *A map $u \in C^2(M, N)$ is a harmonic map iff u satisfies*

$$\Delta_g u^i + g^{\alpha\beta}\Gamma^i_{jk}(u)\frac{\partial u^j}{\partial x_\alpha}\frac{\partial u^k}{\partial x_\beta} = 0 \quad \text{in} \quad M, \ (1 \le i \le l), \tag{1.3}$$

where Δ_g is Laplace-Beltrami operator on (M, g) given by

$$\Delta_g = \frac{1}{\sqrt{g}}\frac{\partial}{\partial x_\alpha}\left(\sqrt{g}g^{\alpha\beta}\frac{\partial}{\partial x_\beta}\right)$$

and

$$\Gamma^i_{jk} = \frac{1}{2}h^{il}(h_{lj,k} + h_{kj,l} - h_{jk,l})$$

is the Christoffel symbol of the metric h on N.

Proof. Let $U \subset M$ be any coordinate chart and $\phi \in C^2_0(U, \mathbb{R}^l)$. Then we have

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=0}\left(\frac{1}{2}\int_M g^{\alpha\beta}h_{ij}(u+t\phi)(u^i_\alpha + t\phi^i_\alpha)(u^j_\beta + t\phi^i_\beta)\sqrt{g}\,dx\right) \\ &= \frac{1}{2}\int_M g^{\alpha\beta}h_{ij,k}(u)\phi^k u^i_\alpha u^j_\beta \sqrt{g}\,dx + \int_M g^{\alpha\beta}h_{ij}(u)u^i_\alpha \phi^j_\beta \sqrt{g}\,dx. \end{aligned}$$

This implies

$$\begin{aligned} \int_M \Delta_g u^i h_{ij}(u)\phi^j\,dv_g &= \frac{1}{2}\int_M g^{\alpha\beta}h_{ij,k}(u)u^i_\alpha u^j_\beta \phi^k\,dv_g \\ &- \int_M g^{\alpha\beta}h_{ij,l}(u)u^i_\alpha u^l_\beta \phi^j\,dv_g. \end{aligned}$$

Choosing $\phi^j = h^{ji}\eta_i$ for $\eta = (\eta_1, \cdots, \eta_l) \in C^2_0(U, \mathbb{R}^l)$, we obtain

$$\begin{aligned} &\int_M \Delta_g u^i \eta^i\,dv_g \\ &= \frac{1}{2}\int_M g^{\alpha\beta}h^{mk}(u)(h_{ij,k}(u) - h_{ik,j}(u) - h_{jk,i}(u))u^i_\alpha u^j_\beta \eta_m\,dv_g. \end{aligned}$$

This yields (1.3). □

1.2 Intrinsic view of harmonic maps

For $u \in C^2(M, N)$, let T^*M be the cotangent bundle of M and u^*TN be the pull-back of the tangent bundle of N by u. View $du = \frac{\partial u^i}{\partial x_\alpha}dx_\alpha \otimes \frac{\partial}{\partial u^i}$ as a section of the bundle $T^*M \otimes u^*TN$. Then $e(u)$ equals to

$$e(u) = \frac{1}{2}\langle du, du\rangle_{T^*M\otimes u^*TN} = \frac{1}{2}\mathrm{tr}_g(u^*h),$$

where $\langle , \rangle_{T^*M\otimes u^*TN}$ denotes the inner product on $T^*M \otimes u^*TN$ induced from T^*M and u^*TN, and u^*h is the pull back of the metric tensor h by u, i.e.,

$$(u^*h)\left(\frac{\partial}{\partial x_\alpha}, \frac{\partial}{\partial x_\beta}\right) = h\left(\frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta}\right) = h_{ij}(u)u^i_\alpha u^j_\beta. \tag{1.4}$$

Let ∇ denote the covariant derivative on $T^*M \otimes u^*TN$ induced from T^*M and u^*TN. Then we have (cf. Eells-Lemaire [50, 51])

Proposition 1.2.1 *$u \in C^2(M,N)$ is a harmonic map iff u satisfies*

$$\tau(u) := tr_g(\nabla du) = 0 \quad \text{in } M. \tag{1.5}$$

Note that

$$\begin{aligned}\nabla_{\frac{\partial}{\partial x_\beta}}(du) &= \nabla_{\frac{\partial}{\partial x_\beta}}\left(u^i_\alpha dx_\alpha \otimes \frac{\partial}{\partial u^i}\right)\\ &= \frac{\partial^2 u^i}{\partial x_\alpha \partial x_\beta} dx_\alpha \otimes \frac{\partial}{\partial u^i} + u^i_\alpha (\nabla^{T^*M}_{\frac{\partial}{\partial x_\beta}} dx_\alpha) \otimes \frac{\partial}{\partial u^i}\\ &+ u^i_\alpha u^j_\beta \left(\nabla^{TN}_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i}\right) \otimes dx_\alpha.\end{aligned}$$

Also

$$\nabla^{TN}_{\frac{\partial}{\partial u^j}} \frac{\partial}{\partial u^i} = (\Gamma^N)^k_{ij}(u) \frac{\partial}{\partial u^k} \qquad \text{and } \nabla^{T^*M}_{\frac{\partial}{\partial x_\beta}} dx_\alpha = -(\Gamma^M)^\alpha_{\beta\gamma}(x)\, dx_\gamma,$$

we conclude that (1.5) is equivalent to

$$\tau^k(u) = g^{\alpha\beta}\left(u^k_{\alpha\beta} - (\Gamma^M)^\gamma_{\alpha\beta} u^k_\gamma + (\Gamma^N)^k_{ij}(u) u^i_\alpha u^j_\beta\right) = 0 \quad \text{in } M,\ 1 \le k \le l. \tag{1.6}$$

1.3 Extrinsic view of harmonic maps

By the isometric embedding theorem by Nash [150], we can assume that (N,h) is isometrically embedded into an Euclidean space $\mathbb{R}^L$ for some $L \ge 1$. Then

$$C^2(M,N) = \left\{u = \left(u^1, \cdots, u^L\right) \in C^2(M, \mathbb{R}^L) \mid u(M) \subset N\right\}.$$

Hence for $u \in C^2(M,N)$ the Dirichlet energy density is

$$e(u) = \frac{1}{2} g^{\alpha\beta} u^i_\alpha u^i_\beta.$$

As $N \subset \mathbb{R}^L$ is a compact smooth submanifold, it is well-known that there exists $\delta = \delta(N) > 0$ such that the nearest point projection map $\Pi_N : N_\delta \to N$ is smooth, where

$$N_\delta = \left\{y \in \mathbb{R}^L \mid d(y,N) := \inf_{z\in N} |y - z| < \delta\right\},$$

and $\Pi_N(y) \in N$ is such that $|y - \Pi_N(y)| = d(y,N)$ for $y \in N_\delta$.

Note that $P(y) = \nabla \Pi_N(y) : \mathbb{R}^L \to T_yN,\ y \in N$, is an orthogonal projection map, and

$$A(y) = \nabla P(y) : T_yN \otimes T_yN \to (T_yN)^\perp,\ y \in N,$$

is the second fundamental form of $N \subset \mathbb{R}^L$.

Now we have

Proposition 1.3.1 *$u \in C^2(M,N)$ is a harmonic map iff u satisfies*

$$\Delta_g u \perp T_uN. \tag{1.7}$$

Proof. For $\phi \in C_0^2(M, \mathbb{R}^L)$, one has

$$\begin{aligned} 0 &= \frac{d}{dt}|_{t=0} \int_M |\nabla (\Pi(u+t\phi))|^2 \, dv_g \\ &= 2\int_M \langle \nabla u, \nabla (P(u)(\phi))\rangle_g \, dv_g \\ &= -2\int_M \langle \Delta_g u, P(u)(\phi)\rangle \, dv_g \\ &= -2\int_M \langle P(u)(\Delta_g u), \phi\rangle \, dv_g. \end{aligned}$$

This clearly implies (1.7). □

Let $\{\nu_{l+1}(u), \cdots, \nu_L(u)\}$ be a local orthonormal frame of the normal bundle $(T_u N)^\perp$. Then (1.7) implies

$$\Delta_g u = \sum_{l+1 \le i \le L} \lambda_i(x)\nu_i(u)$$

for some functions $(\lambda_{l+1}, \cdots, \lambda_L)$ on M. Moreover, for $l+1 \le i \le L$,

$$\begin{aligned} \lambda_i &= \Delta_g u \cdot \nu_i(u) \\ &= \operatorname{div}_g (\nabla u \cdot \nu_i(u)) - \nabla u \cdot \nabla (\nu_i(u)) \\ &= -(\nabla \nu_i)(u)(\nabla u, \nabla u) \end{aligned}$$

where we have used $\nabla u \cdot \nu_i(u) = 0$, and div_g is the divergence operator on (M, g) given by

$$\operatorname{div}_g = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x_\alpha} \left(\sqrt{g} g^{\alpha\beta}\right).$$

Therefore we obtain the analytic version of (1.7):

$$\Delta_g u + A(u)(\nabla u, \nabla u) = 0 \text{ in } M, \tag{1.8}$$

where

$$A(u)(\nabla u, \nabla u) = \sum_{l+1 \le i \le L} g^{\alpha\beta} A^i(u)(u_\alpha, u_\beta)\, \nu_i(u),$$

and $A^i = \nabla \nu_i$ is the second fundamental form of N in the normal direction ν_i.

Example 1.3.2 Let $M = T^n$ be the n-dimensional torus, and $N = S^k \subset \mathbb{R}^{k+1}$ be the unit sphere. Then $u \in C^2(T^n, S^k)$ is a harmonic map iff

$$\Delta u + |\nabla u|^2 u = 0 \text{ in } T^n. \tag{1.9}$$

1.4 A few facts about harmonic maps

Proposition 1.4.1 *If $\Phi : M \to M$ is a C^2-diffeomorphism and $u \in C^2(M, N)$ is a harmonic map with respect to (M, g), then $u \circ \Phi \in C^2(M, N)$ is a harmonic map with respect to $(M, \Phi^* g)$.*

Proof. This is an easy consequence of the identity:

$$\int_M |\nabla v|_g^2 \, dv_g = \int_M |\nabla (v \circ \Phi)|_{\Phi^* g}^2 \, dv_{\Phi^* g}$$

for all $v \in C^2(M, N)$. □

Proposition 1.4.2 *Let (M, g_1) be a Riemannian surface, $\Phi : (M, g_1) \to (M, g_2)$ be a conformal map. If $u \in C^2(M, N)$ is a harmonic map with respect to (M, g_2), then $u \circ \Phi \in C^2(M, N)$ is a harmonic map with respect to (M, g_1).*

Proof. This follows from the conformal invariance of the Dirichlet energy functional E in dimension two. In fact, let $\phi \in C^2(M)$ be such that $\Phi^* g_2 = e^{2\phi} g_1$. Then we have, for any $v \in C^2(M, N)$,

$$\begin{aligned} E(v \circ \Phi, g_1) &= \frac{1}{2} \int_M \mathrm{tr}_{g_1} \left((v \circ \Phi)^* h \right) dv_{g_1} \\ &= \frac{1}{2} \int_M \mathrm{tr}_{e^{-2\phi} \Phi^* g_2} \left(\Phi^* \circ v^* h \right) e^{-2\phi} \, dv_{\Phi^* g_2} \\ &= \frac{1}{2} \int_M \mathrm{tr}_{\Phi^* g_2} \left(\Phi^* \circ (v^* h) \right) dv_{\Phi^* g_2} \\ &= \frac{1}{2} \int_M \mathrm{tr}_{g_2} (v^* h) \, dv_{g_2} = E(v, g_2). \end{aligned}$$

This completes the proof of Proposition 1.4.2. □

Remark 1.4.3 (a) Harmonic maps from S^1 to N correspond to closed geodesics in N.
(b) The set of harmonic maps from a Riemannian surface M depends only on the conformal structures of M.
(c) Let $\mathrm{Id} : (M, g) \to (M, g)$ be the identity map. Then Id is a harmonic map.
(d) For $n = \dim(M) = 2$, any conformal map $\phi : (M, g_1) \to (M, g_2)$ is a harmonic map.

Proof. We only indicate the proof of (c). Denote $u(x) = \mathrm{Id}(x) = x$. Then we have

$$\begin{aligned} \tau^k(u) &= g^{\alpha\beta} \left(u^k_{\alpha\beta} - (\Gamma^M)^\gamma_{\alpha\beta} u^k_\gamma + (\Gamma^N)^k_{ij}(u) u^i_\alpha u^j_\beta \right) \\ &= g^{\alpha\beta} \left(0 - (\Gamma^M)^\gamma_{\alpha\beta} \delta_{k\gamma} + (\Gamma^M)^k_{ij}(u) \delta_{i\alpha} \delta_{i\beta} \right) = 0 \end{aligned}$$

for $1 \le k \le n$. □

1.5 Bochner identity for harmonic maps

One of the most important formulas for a harmonic map $u : M \to N$ is the differential equation satisfied by the energy density $e(u)$.

Denote by R^M and R^N the Riemannian curvature tensors of M and N respectively, Ric^M be the Ricci curvature of M. For $x_0 \in M$, in a local coordinate system centered at x_0, write

$$R^M = (R_{\alpha\beta\gamma\delta}), \ \mathrm{Ric}^M = (R_{\alpha\beta}), \ \text{and } R^N = (\hat{R}_{ijkl}),$$

and K^N denotes the sectional curvature of N

Theorem 1.5.1 *If $u \in C^2(M,N)$ is a harmonic map, then in a local coordinate system it holds*

$$\Delta_g e(u) = |\nabla du|^2 + R_{\alpha\beta}(u_\alpha, u_\beta) - \hat{R}_{ijkl}(u)\left(u^i_\alpha, u^j_\beta, u^k_\alpha, u^l_\beta\right) \tag{1.10}$$

where ∇ denotes the covariant derivative on $T^*M \otimes u^*TN$.

Proof. For $x_0 \in M$, let (x_α) be the normal coordinate system centered at x_0. Assume that (N,h) is isometrically embedded in $\mathbb{R}^L$. Then we have

$$\begin{aligned}\Delta_g e(u) &= |u_{\alpha\beta}|^2 + \langle u_\alpha, u_{\beta\alpha,\beta}\rangle \\ &= |u_{\alpha\beta}|^2 + \langle u_\alpha, u_{\beta\beta,\alpha}\rangle + R_{\alpha\beta}(u_\alpha, u_\beta) \\ &= |u_{\alpha\beta}|^2 + \langle u_\alpha, (\Delta_g u)_\alpha\rangle + R_{\alpha\beta}(u_\alpha, u_\beta)\end{aligned}$$

where we have used the Ricci identity

$$u_{\beta\alpha,\beta} = u_{\beta\beta,\alpha} + R_{\alpha\beta}u_\beta.$$

On the other hand, since u is harmonic map, (1.8) implies

$$\begin{aligned}\langle u_\alpha, (\Delta_g u)_\alpha\rangle &= -\langle u_\alpha, (A(u)(\nabla u, \nabla u))_\alpha\rangle \\ &= \langle \Delta_g u, A(u)(\nabla u, \nabla u)\rangle \\ &= -\langle A(u)(\nabla u, \nabla u), A(u)(\nabla u, \nabla u)\rangle \\ &= -\langle A(u)(u_\alpha, u_\alpha), A(u)(u_\beta, u_\beta)\rangle\end{aligned}$$

where we have used the fact that

$$\langle u_\alpha, A(u)(\nabla u, \nabla u)\rangle = 0.$$

For $u_{\alpha\beta}$, it is easy to see that

$$|u_{\alpha\beta}|^2 = |P(u)(u_{\alpha\beta})|^2 + |A(u)(u_\alpha, u_\beta)|^2 = |\nabla du|^2 + |A(u)(u_\alpha, u_\beta)|^2.$$

Putting all these identities together, we obtain

$$\begin{aligned}\Delta_g e(u) &= |\nabla du|^2 + \mathrm{Ric}^M(\nabla u, \nabla u) \\ &- \left\{\langle A(u)(u_\alpha, u_\alpha), A(u)(u_\beta, u_\beta)\rangle - |A(u)(u_\alpha, u_\beta)|^2\right\}\end{aligned}$$

This, with the help of Gauss-Kodazi equation (see [63, 175]):

$$\langle R^N(u)(X,Y)X,Y\rangle = \langle A(u)(X,X), A(u)(Y,Y)\rangle - |A(u)(X,Y)|^2, \ \forall X,Y \in T_uN$$

yields (1.10). □

Proposition 1.5.2 *If (M,g) is compact without boundary with $Ric^M \geq 0$ and the sectional curvature of N, K^N, is non-positive. Then any harmonic map $u \in C^2(M,N)$ is totally geodesic. If $Ric^M > 0$ at a point in M, then u is constant. If $K^N < 0$, then either u is constant or $u(M)$ is contained in a closed geodesic.*

Proof. It follows from (1.10) that $e(u)$ is a subharmonic function on M. Hence the maximum principle implies $e(u) =$ constant and hence $|\nabla du| = 0$. This says that u is totally geodesic.
If $Ric^M(p_0) > 0$, then $\nabla u(p_0) = 0$ and hence $e(u) \equiv 0$ and u is constant map.
If $K^N < 0$, then the linear space span $\{u_1, \cdots, u_n\}$ is at most dimension one. Hence either u is constant or the image of u lies inside a geodesic. □

1.6 Second variational formula of harmonic maps

In this section, we derive the second variational formula for harmonic maps into spheres and general target manifolds.

Proposition 1.6.1 *If $u \in C^2(M, S^k)$ is a harmonic map and $\phi \in C_0^2(M, \mathbb{R}^{k+1})$, then*

$$\frac{d^2}{dt^2}|_{t=0} \left(\frac{1}{2}\int_M |\nabla(\frac{u+t\phi}{|u+t\phi|}|^2 \, dv_g\right) = \int_M \left(|\nabla\hat{\phi}|^2 - |\nabla u|^2|\hat{\phi}|^2\right) dv_g, \qquad (1.11)$$

where $\hat{\phi}(\equiv \phi - \langle u, \phi\rangle u)$ is the tangential component of ϕ.

Proof. For $\phi \in C_0^\infty(M, \mathbb{R}^{k+1})$ and small $t \in \mathbb{R}$, denote $u_t = \frac{u+t\phi}{|u+t\phi|}$. Then direct calculations give

$$\frac{du_t}{dt}|_{t=0} = \phi - \langle u, \phi\rangle u = \hat{\phi},$$

and

$$\frac{d^2u_t}{dt^2}|_{t=0} = 3\langle u, \phi\rangle^2 u - |\phi|^2 u - 2\langle u, \phi\rangle\phi.$$

Hence we have

$$\begin{aligned}
&\frac{d^2}{dt^2}|_{t=0} \left(\frac{1}{2}\int_M |\nabla(\frac{u+t\phi}{|u+t\phi|})|^2 \, dv_g\right) \\
= &\int_M \left(|\nabla(\frac{du_t}{dt}|_{t=0})|^2 + \langle\nabla u, \nabla(\frac{d^2u_t}{dt^2}|_{t=0})\rangle\right) dv_g \\
= &\int_M \left(|\nabla\hat{\phi}|^2 - \langle\Delta_g u, 3\langle u,\phi\rangle^2 u - |\phi|^2 u - 2\langle u,\phi\rangle\phi\rangle\right) dv_g \\
= &\int_M \left(|\nabla\hat{\phi}|^2 + \langle|\nabla u|^2 u, 3\langle u,\phi\rangle^2 u - |\phi|^2 u - 2\langle u,\phi\rangle\phi\rangle\right) dv_g \\
= &\int_M \left(|\nabla\hat{\phi}|^2 - |\nabla u|^2(|\phi|^2 - |\langle u,\phi\rangle|^2)\right) dv_g \\
= &\int_M \left(|\nabla\hat{\phi}|^2 - |\nabla u|^2|\hat{\phi}|^2\right) dv_g.
\end{aligned}$$

This completes the proof of (1.11). □

Next we derive a general second variational formula for the Dirichlet energy functional.

Proposition 1.6.2 *Let $u \in C^2(M,N)$ be a harmonic map, and $u_t \in C^2([0,1] \times M, N)$ be a family of smooth variations of u, i.e., $u_0 = u$. Let $v = \frac{du_t}{dt}|_{t=0} \in C^2(M, u^*TN)$. Then*

$$\frac{d^2}{dt^2}|_{t=0}\left(\frac{1}{2}\int_M |\nabla u_t|_g^2\, dv_g\right) \tag{1.12}$$

$$= \int_M \left(|\nabla v|_g^2 - \mathrm{tr}_g\langle R^N(v,\nabla u)\, v, \nabla u\rangle\right) dv_g. \tag{1.13}$$

In particular, if $K^N \le 0$, then u is stable, i.e.,

$$\frac{d^2}{dt^2}|_{t=0}\left(\frac{1}{2}\int_M |\nabla u_t|_g^2\, dv_g\right) \ge 0.$$

Proof. Let $(\frac{\partial}{\partial x_\alpha})$ be a local coordinate frame on M. Then we have

$$\frac{d}{dt}|_{t=0}\frac{\partial u_t}{\partial x_\alpha} = \nabla^{u^*TN}_{\frac{\partial}{\partial t}}(\frac{\partial u_t}{\partial x_\alpha}) = \nabla^{u^*TN}_{\frac{\partial}{\partial x_\alpha}} v,$$

as $[\frac{\partial u_t}{\partial t}, \frac{\partial u_t}{\partial x_\alpha}] = 0$. Hence, we have

$$\begin{aligned}\frac{d^2}{dt^2}|_{t=0}\frac{\partial u_t}{\partial x_\alpha} &= \nabla^{u^*TN}_{\frac{\partial}{\partial t}}\nabla^{u^*TN}_{\frac{\partial}{\partial x_\alpha}} v \\ &= \nabla^{u^*TN}_{\frac{\partial}{\partial x_\alpha}}\nabla^{u^*TN}_{\frac{\partial}{\partial t}} v + R^N(v, \frac{\partial u}{\partial x_\alpha})v.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\frac{d^2}{dt^2}\left(\frac{1}{2}\int_M |\nabla u_t|_g^2\, dv_g\right)\\ = &\int_M \left(|\nabla v|_g^2 + \langle \nabla u, \nabla\left(\frac{d^2 u_t}{dt^2}|_{t=0}\right)\rangle_g\right) dv_g\\ = &\int_M \left(|\nabla v|_g^2 + \langle \frac{\partial u}{\partial x_\alpha}, \nabla^{u^*TN}_{\frac{\partial}{\partial x_\alpha}}(\nabla^{u^*TN}_{v} v)\rangle - \mathrm{tr}_g(R^N(v,\nabla u)v, \nabla u)\right) dv_g\\ = &\int_M \left(|\nabla v|_g^2 - \langle \tau(u), \nabla^{u^*TN}_{v} v\rangle - \mathrm{tr}_g\left(R^N(v,\nabla u)v, \nabla u\right)\right) dv_g.\end{aligned}$$

Since $\tau(u) = 0$, this implies (1.13). If $K^N \le 0$, then we can easily conclude that u is stable. □

Chapter 2

Regularity of minimizing harmonic maps

In this chapter, we will present the regularity theorems of minimizing harmonic maps between Riemannian manifolds. This includes (i) the regularity theorem of minimizing harmonic maps in dimensions two by Morrey [145], (ii) the partial regularity theorem of minimizing harmonic maps in dimensions at least three by Schoen-Uhlenbeck [171] and Giaquinta-Giusti [66], (iii) Federer's dimension reduction principle for minimizing harmonic maps by [171] (see also [54]), and (iv) the uniqueness theorem on minimizing tangent maps by Simon [181, 182, 183].

2.1 Minimizing harmonic maps in dimension two

First recall that the usual Sobolev space $W^{1,2}(M,\mathbb{R}^L)$ consists of any $\mathbb{R}^L$-valued function $u = (u^1, \cdots, u^L) \in L^2(M)$ such that its distributional derivative $\nabla u = (\frac{\partial u}{\partial x_1}, \cdots, \frac{\partial u}{\partial x_n}) \in L^2(M)$.
The Sobolev space, $W^{1,2}(M,N)$, of maps from M to N is defined by

$$\begin{aligned} W^{1,2}(M,N) &= \{v : M \to \mathbb{R}^L \mid \|v\|^2_{W^{1,2}(M)} = \int_M \left(|v|^2 + e_g(v)\right) dv_g < +\infty \\ &\quad v(x) \in N \text{ for a.e. } x \in M\}. \end{aligned}$$

Note that $W^{1,2}(M,N) \subset W^{1,2}(M,\mathbb{R}^L)$ is closed under sequentially weak convergence in $W^{1,2}(M,\mathbb{R}^L)$.
It is worthwhile to note that, in contrast with $W^{1,2}(M,\mathbb{R}^L)$, $C^\infty(M,N)$ may not be dense in $W^{1,2}(M,N)$. This is illustrated by the following example, due to Schoen-Uhlenbeck [173].

Remark 2.1.1 $\phi(x) = \frac{x}{|x|} \in W^{1,2}(B^3, S^2)$ can't be approximated by smooth maps from B^3 to S^2 in $W^{1,2}(B^3, S^2)$.

Proof. Note that

$$\int_{B^3} \left|\nabla(\frac{x}{|x|})\right|^2 = \int_0^1 \int_{S^2} \frac{1}{r^2} \left|\nabla_{S^2}\left(\frac{x}{|x|}\right)\right|^2 r^2\, dH^2 dr = 2H^2(S^2) = 8\pi.$$

Suppose that there exist $\{u_i\} \subset C^\infty(B^3, S^2)$ such that

$$\lim_{i\to\infty} \|\nabla (u_i - \phi)\|_{L^2(B^3)} = 0.$$

By Fatou's Lemma and Fubini's theorem, this implies that there exists $r \in (\frac{1}{2}, 1)$ such that

$$\lim_{i\to\infty} \int_{\partial B_r} |\nabla (u_i - \phi)|^2 \, dH^2 = 0.$$

Recall that for any $f \in C^1(\partial B_r, S^2)$, the topological degree of f is given by

$$\deg(f; \partial B_r) = \frac{1}{4\pi} \int_{\partial B_r} \det(\nabla_T f) \, dH^2 \tag{2.1}$$

where ∇_T denotes the tangential gradient on ∂B_r.
Since $u_i \in C^\infty(B^3, S^2)$, $\deg(u_i; \partial B_r) = 0$ for any $0 < r \le 1$. On the other hand, since $\deg(\phi; \partial B_r) = 1$ and $u_i \to \phi$ in $W^{1,2}(\partial B_r, S^2)$, we have

$$\lim_{i\to\infty} \frac{1}{4\pi} \int_{\partial B_r} \det(\nabla_T u_i) \, dH^2 = \frac{1}{4\pi} \int_{\partial B_r} \det(\nabla_T \phi) = 1.$$

This gives a contradiction and completes the proof. □

On the other hand, we have

Remark 2.1.2 For $\dim(M) = 2$, $C^\infty(M, N)$ is dense in $W^{1,2}(M, N)$. For $n \ge 3$, if $\Pi_2(N) = \{0\}$ then $C^\infty(B^n, N)$ is dense in $W^{1,2}(B^n, N)$.

The proof of the first part can be found in [173], and the second part can be found in Bethuel [10] and Hang-Lin [79].

Now we introduce the notion of weakly harmonic maps and minimizing harmonic maps.

Definition 2.1.3 A map $u \in W^{1,2}(M, N)$ is weakly harmonic map, if it satisfies the harmonic map equation (1.8) in the sense of distributions, i.e.,

$$\int_M g^{\alpha\beta} \left(\langle \frac{\partial u}{\partial x_\alpha}, \frac{\partial \phi}{\partial x_\beta} \rangle + A(u)(\frac{\partial u}{\partial x_\alpha}, \frac{\partial u}{\partial x_\beta})\phi \right) dv_g = 0 \tag{2.2}$$

for any $\phi \in C_0^\infty(M, \mathbb{R}^L)$.

One of the most important classes of weakly harmonic maps is the class of minimizing harmonic maps.

Definition 2.1.4 A map $u \in W^{1,2}(M, N)$ is a minimizing harmonic map, if for any $\Omega \subset M$,

$$E(u, \Omega) := \frac{1}{2} \int_\Omega e(u) \, dv_g \le E(v, \Omega) \tag{2.3}$$

for any $v \in W^{1,2}(\Omega, N)$ with $v|_{\partial\Omega} = u|_{\partial\Omega}$.

We immediately have the following proposition.

Proposition 2.1.5 *Any minimizing harmonic map $u \in W^{1,2}(M,N)$ is a weakly harmonic map.*

Proof. For any $\Omega \subset M$, let $\phi \in C_0^1(\Omega, \mathbb{R}^L)$ and $t \in \mathbb{R}$ be sufficiently small. Then we have

$$E(u,\Omega) \le E\left(\Pi_N\left(u + t\phi\right), \Omega\right).$$

Hence

$$\frac{d}{dt}|_{t=0} \int_\Omega e\left(\Pi_N(u + t\phi)\right)\, dv_g = 0.$$

Then direct calculations imply u satisfies (2.3). □

The existence of minimizing harmonic maps can be obtained by the direct method in calculus of variations.

Proposition 2.1.6 *For $\partial M \neq \emptyset$, suppose that $\phi \in W^{1,2}(M,N)$ is given. Then there exists at least one minimizing harmonic map $u \in W^{1,2}(M,N)$ with $u|_{\partial M} = \phi|_{\partial M}$.*

Proof. Set $\mathcal{A} = \left\{v \in W^{1,2}(M,N) \mid v|_{\partial M} = \phi|_{\partial M}\right\}$ and define

$$c := \inf\left\{E(v) \mid v \in \mathcal{A}\right\}.$$

Since $\phi \in \mathcal{A}$, $c \le E(\phi) < +\infty$. Let $\{u_i\} \subset \mathcal{A}$ be a minimizing sequence, i.e.,

$$\lim_{i\to\infty} \frac{1}{2}\int_M e(u_i)\, dv_g = c.$$

Then $\{u_i\} \subset W^{1,2}(M,N)$ is a bounded sequence. By the weak compactness, there exist a subsequence $\{u_{i'}\}$ of $\{u_i\}$ and a map $u \in W^{1,2}(M,N)$ such that $u_{i'} \to u$ weakly in $W^{1,2}(M,N)$. By the Rellich's compactness theorem and the trace theorem, we have

$$\lim_{i'\to\infty}\left(\|u_{i'} - u\|_{L^2(M)} + \|u_{i'} - u\|_{L^2(\partial M)}\right) = 0.$$

Hence $u|_{\partial M} = \phi|_{\partial M}$, $u \in \mathcal{A}$ and $E(u) \ge c$. On the other hand, by the lower semicontinuity, we have

$$\frac{1}{2}\int_M e(u)\, dv_g \le \frac{1}{2}\lim_{i'\to\infty}\int_M e(u_i)\, dv_g = c.$$

Therefore we have $E(u) = c$ and u is a minimizing harmonic map. □

The regularity issue of minimizing harmonic maps has been studied extensively by many people in the last several decades. The first result seems to go back to Morrey [145].

Theorem 2.1.7 *For $\dim(M) = 2$ if $u \in W^{1,2}(M,N)$ is a minimizing harmonic map, then $u \in C^\infty(M,N)$.*

Proof. Since the regularity is of local nature, we may assume $(M,g)=(B,g_0)$, the unit ball in $\mathbb{R}^2$ with the standard metric. By the absolute continuity of $\int |\nabla u|^2\,dx$, we have that for any $\epsilon>0$, there exists $r_0>0$ such that

$$\int_{B_r(x)} |\nabla u|^2\,dx \le \epsilon^2,\ \forall x\in B_{\frac{1}{2}},\ r\le r_0. \tag{2.4}$$

By Fubini's theorem, there exist $r_1\in(\frac{r}{2},r)$ such that

$$r_1\int_{\partial B_{r_1}(x)} |\nabla u|^2\,dH^1 \le 8\int_{B_r(x)\setminus B_{\frac{r}{2}}(x)} |\nabla u|^2 (\le 8\epsilon^2). \tag{2.5}$$

By the Sobolev embedding theorem, $u\in W^{1,2}(\partial B_{r_1}(x))\subset C^{\frac{1}{2}}(\partial B_{r_1}(x))$ and

$$|u(y)-u(z)|\le C\epsilon\left(\frac{|y-z|}{r_1}\right)^{\frac{1}{2}},\ \forall y,z\in\partial B_{r_1}(x). \tag{2.6}$$

Therefore there exists $p_0\in N$ such that

$$u(\partial B_{r_1}(x))\subset B^L_{C\epsilon}(p_0)\cap N,$$

where $B^L_{C\epsilon}(p_0)\subset\mathbb{R}^L$ is the L-dimensional ball with center p_0 and radius $C\epsilon$.
Let $v\in W^{1,2}(B_{r_1},\mathbb{R}^L)$ be the solution to

$$\begin{aligned}\Delta v &= 0 \text{ in } B_{r_1}(x)\\ v &= u \text{ on } \partial B_{r_1}(x).\end{aligned}$$

The maximum principle and (2.6) imply that $v(B_{r_1}(x))\subset B^L_{C\epsilon}(p_0)$. Hence for ϵ sufficiently small, we have $v(B_{r_1}(x))\subset N_\delta$. By the minimality, we then have

$$\int_{B_{r_1}(x)} |\nabla u|^2 \le \int_{B_{r_1}(x)} |\nabla(\Pi_N(v))|^2 \le C\int_{B_{r_1}(x)} |\nabla v|^2. \tag{2.7}$$

On the other hand, by the standard estimate for harmonic functions, we have

$$\int_{B_{r_1}(x)} |\nabla v|^2 \le Cr_1\int_{\partial B_{r_1}(x)} |\nabla u|^2\,dH^1. \tag{2.8}$$

Putting these inequalities together gives

$$\int_{B_{\frac{r}{2}}(x)} |\nabla u|^2 \le C\int_{B_r(x)\setminus B_{\frac{r}{2}}(x)} |\nabla u|^2.$$

Hence

$$\int_{B_{\frac{r}{2}}(x)} |\nabla u|^2 \le \theta\int_{B_r(x)} |\nabla u|^2 \tag{2.9}$$

where $0<\theta=\frac{C}{C+1}<1$.
Iterating (2.9) k-times, we get

$$\int_{B_{2^{-k}r}(x)} |\nabla u|^2 \le \theta^k\int_{B_r(x)} |\nabla u|^2,\ \forall x\in B_{\frac{1}{2}} \text{ and } 0<r\le r_0. \tag{2.10}$$

Set $\alpha = \frac{|\ln\theta|}{2\ln 2} \in (0,1)$. Then (2.10) implies

$$\int_{B_r(x)} |\nabla u|^2 \le C\left(\frac{r}{r_0}\right)^{2\alpha} \int_{B_{r_0}(x)} |\nabla u|^2, \ \forall x \in B_{\frac{1}{2}} \text{ and } 0 < r \le r_0. \tag{2.11}$$

This, combined with lemma 2.1.10 below, implies $u \in C^\alpha(B_{\frac{1}{2}}, N)$. Then theorem 2.1.8 yields that $u \in C^{1,\beta}(B_{\frac{1}{2}}, N)$ for some $\beta \in (0,1)$. The higher order regularity of u then follows from the standard theory of linear elliptic equations (see Gilbarg-Trudinger [72]). □

We now want to show $C^{1,\beta}$-regularity for u.

Theorem 2.1.8 *For $n \ge 2$ and $\alpha \in (0,1)$, let $B \subset \mathbb{R}^n$ be a unit ball and $u \in W^{1,2}(B,N) \cap C^\alpha(B,N)$ be a weakly harmonic map satisfying, $\forall x \in B_{\frac{1}{2}}$ and $0 < r \le r_0$,*

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2 \le C\left(\frac{r}{r_0}\right)^{2\alpha} r_0^{2-n} \int_{B_{r_0}(x)} |\nabla u|^2. \tag{2.12}$$

Then there exists $\beta \in (0,1)$ such that $u \in C^{1,\beta}(B_{\frac{1}{2}}, N)$.

Before proving this theorem, we recall the characterization of the Hölder space, C^α, due to Campanato [25].

Proposition 2.1.9 *For $\alpha \in (0,1)$ and an open set $E \subset \mathbb{R}^n$, define*

$$\mathcal{C}^\alpha(E) = \left\{ f \in L^1(E) \mid [f]_{\mathcal{C}^\alpha(E)} = \sup_{B_r \subset E} r^{-(n+\alpha)} \int_{B_r} |f - f_{B_r}| < +\infty \right\}$$

where $f_{B_r} = \frac{1}{|B_r|}\int_{B_r} f$ is the average of f over B_r. Then $\mathcal{C}^\alpha(E) = C^\alpha(E)$ and there exists $C > 0$ such that

$$C^{-1}[f]_{\mathcal{C}^\alpha(E)} \le [f]_{C^\alpha(E)} \le C[f]_{\mathcal{C}^\alpha(E)}. \tag{2.13}$$

By the Poincaré inequality, Proposition 2.1.9 implies Morrey's decay Lemma (see also [145]).

Lemma 2.1.10 *For $\alpha \in (0,1)$ and any open set $B \subset \mathbb{R}^n$, if $f : B \to \mathbb{R}$ satisfies*

$$\|\nabla f\|^2_{M^{2,n+2\alpha-2}(B)} := \sup_{B_r(x) \subset B} \left\{ r^{2(1+\alpha)-n} \int_{B_r(x)} |\nabla f|^2 \right\} < +\infty \tag{2.14}$$

then $u \in C^\alpha(B)$ and

$$[u]_{C^\alpha(B)} \le C\|\nabla f\|_{M^{2,2+2\alpha-n}(B)}. \tag{2.15}$$

Proof of Theorem 2.1.8:

Step 1. $u \in C^\delta(B_{\frac{1}{2}}, N)$ for any $\delta \in (0,1)$.

Proposition 2.2.3 and (2.2.10) imply that for any $0 < r \le r_0$ and $x \in B^n_{\frac{1}{2}}$,

$$[u]^2_{C^\alpha(B_r(x))} \le C r_0^{2-2\alpha-n} \int_{B_{r_0}(x)} |\nabla u|^2. \tag{2.16}$$

Let $v \in W^{1,2}(B_r(x), \mathbb{R}^L)$ solve

$$\begin{aligned} \Delta v &= 0 \quad \text{in} \quad B_r(x) \\ v &= u \quad \text{on} \quad \partial B_r(x). \end{aligned} \tag{2.17}$$

Then we have

$$\operatorname{osc}_{B_r(x)} v \leq \operatorname{osc}_{B_r(x)} u \leq C r^\alpha.$$

Subtracting (2.2) from (2.17), multiplying $(u - v)$, and integrating the resulting equation over $B_r(x)$, we have

$$\begin{aligned} r^{2-n} \int_{B_r(x)} |\nabla (u-v)|^2 &\leq C \operatorname{osc}(u-v)\, r^{2-n} \int_{B_r(x)} |\nabla u|^2 \\ &\leq C r^{3\alpha} \left(r_0^{2-2\alpha-n} \int_{B_{r_0}(x)} |\nabla u|^2 \right)^{\frac{3}{2}}. \end{aligned}$$

Hence for any $\theta \in (0, \frac{1}{2})$,

$$\begin{aligned} &(\theta r)^{2-n} \int_{B_{\theta r}(x)} |\nabla u|^2 \\ &\leq 2 \left((\theta r)^{2-n} \int_{B_{\theta r}(x)} |\nabla v|^2 + \theta^{2-n} r^{2-n} \int_{B_r(x)} |\nabla (u-v)|^2 \right) \\ &\leq C \left(\theta^2 r^{2+2\alpha} + \theta^{2-n} r^{3\alpha} \right) \left(r_0^{2-n-2\alpha} \int_{B_{r_0}(x)} |\nabla u|^2 \right) \\ &\quad \cdot \left(1 + (r_0^{2-n-2\alpha} \int_{B_{r_0}(x)} |\nabla u|^2)^{\frac{1}{2}} \right) \end{aligned}$$

where we have used (2.12) and the estimate for v:

$$(\theta r)^{2-n} \int_{B_{\theta r}(x)} |\nabla v|^2 \leq C (\theta r)^2 r^{2-n} \int_{B_r(x)} |\nabla u|^2.$$

This implies $u \in C^\delta(B_{\frac{1}{2}}, N)$ for any $\alpha \leq \delta < 1$.
Step 2. $u \in C^{1,\beta}(B^n_{\frac{1}{2}}, N)$ for some $\beta \in (0,1)$.
By Proposition 2.1.9, it suffices to show

$$r^{-n} \int_{B_r(x)} |\nabla u - (\nabla u)_{x,r}|^2 \leq C r^{2\beta}, \ \forall x \in B_{\frac{1}{2}}, \ 0 < r \leq r_0, \tag{2.18}$$

where $(\nabla u)_{x,r}$ is the average of ∇u on $B_r(x)$.
In fact, for any $\theta \in (0, \frac{1}{2})$ we have

$$
\begin{aligned}
&(\theta r)^{-n} \int_{B_{\theta r}(x)} |\nabla u - (\nabla u)_{x,\theta r}|^2 \\
&\leq (\theta r)^{-n} \int_{B_{\theta r}(x)} |\nabla u - (\nabla v)_{x,\theta r}|^2 \\
&\leq 2\left((\theta r)^{-n} \int_{B_{\theta r}(x)} |\nabla v - (\nabla v)_{x,\theta r}|^2 + (\theta r)^{-n} \int_{B_r(x)} |\nabla (u-v)|^2 \right) \\
&\leq C\left(r^2 + \theta^{-n} r^{3\delta - 2}\right)\left(r^{2-n}\int_{B_r(x)} |\nabla u|^2\right)\left(1 + (r^{2-n}\int_{B_r(x)} |\nabla u|^2)^{\frac{1}{2}}\right) \\
&\leq C r^{2\beta},
\end{aligned}
$$

for $\beta = \min\{1, \frac{3\delta-2}{2}\}$ if we choose $\delta > \frac{2}{3}$, where we have used the estimate on v:

$$
(\theta r)^{-n} \int_{B_{\theta r}(x)} |\nabla v - (\nabla v)_{x,\theta r}|^2 \leq C\theta^2 r^{-n} \int_{B_r(x)} |\nabla v - (\nabla v)_{x,r}|^2 .
$$

This implies (2.18). Hence by proposition 2.1.9, we have $u \in C^{1,\beta}(B_{\frac{1}{2}}, N)$ for some $\beta \in (0,1)$. □

2.2 Minimizing harmonic maps in higher dimensions

Regularity for minimizing harmonic maps in dimensions at least three is much more delicate. First, concrete examples indicate that minimizing harmonic maps in higher dimensions may have singularities. More precisely,

Proposition 2.2.1 *For $n \geq 3$, $\phi_0(x) = \frac{x}{|x|} : B^n \to S^{n-1}$ is a minimizing harmonic map.*

Remark 2.2.2 Proposition 2.2.1 was first proved by Jäger-Kaul [108] for $n \geq 7$, later by Brezis-Coron-Lieb [19] for $n = 3$, and finally by Lin [122] and Coron-Gulliver [40] independently for all $n \geq 3$.

Proof. We present the proof by [122]. First we claim that

$$
|\nabla \phi|^2 \geq \frac{1}{n-2}\left((\mathrm{div}\phi)^2 - \mathrm{tr}\,(\nabla\phi)^2\right) \quad \text{a.e. in } B^n \tag{2.19}
$$

for any $\phi \in W^{1,2}(B^n, S^{n-1})$, where $(\nabla\phi)^2$ is the square of $\nabla\phi$.
To prove (2.19), we make a change of coordinates so that at $x_0 \in B^n$,

$$
\phi(x_0) = (0, \cdots, 1).
$$

For $1 \leq i, j \leq n$, set $a_{ij} = \frac{\partial \phi^i}{\partial x_j}(x_0)$. Since $|\phi|^2 = 1$, we have $a_{nj} = 0$ for $1 \leq j \leq n$ so that

$$
|\nabla\phi|^2(x_0) = \sum_{1\leq i,j\leq n} a_{ij}^2 = \sum_{1\leq i\leq n-1} a_{ii}^2 + \sum_{i\neq j} a_{ij}^2,
$$

$$\left(\mathrm{div}\phi\right)^2(x_0) = \left(\sum_{1\le i\le n-1} a_{ii}\right)^2 \le (n-1)\sum_{1\le i\le n-1} a_{ii}^2,$$

and

$$\mathrm{tr}\left(\nabla\phi\right)^2(x_0) = \sum_{1\le i,j\le n} a_{ij}a_{ji} \ge \sum_{1\le i\le n-1} a_{ii}^2 - \sum_{i\ne j} a_{ij}^2.$$

Putting these inequalities together, we obtain (2.19).
On the other hand, direct calculations imply that

$$\left(\mathrm{div}\phi\right)^2 - \mathrm{tr}\left(\nabla\phi\right)^2 = \mathrm{div}\left(\left(\mathrm{div}\phi\right)\phi - \left(\nabla\phi\right)\phi\right).$$

Hence integrating (2.19) over B^n leads

$$\begin{aligned}
\int_{B^n} |\nabla\phi|^2 \quad &\ge \frac{1}{n-2}\int_{B^n} (\mathrm{div}\ \phi)^2 - \mathrm{tr}\left(\nabla\phi\right)^2 \\
&= \frac{1}{n-2}\int_{\partial B^n} \{(\mathrm{div}\phi)\phi\cdot x - (\nabla\phi)\phi\cdot x\} \\
&= \frac{1}{n-2}\int_{\partial B^n} \{(\mathrm{div}x)x\cdot x - (\nabla x)x\cdot x\} \\
&= \frac{1}{n-2}\int_{\partial B^n} (n-1) = \frac{n-1}{n-2}\left|S^{n-1}\right| = \int_{B^n} |\nabla\phi_0|^2.
\end{aligned}$$

This completes the proof. □

A slight modification of this example shows that a minimizing harmonic map from $\mathbb{R}^n$, $n\ge 3$, can have $(n-3)$-dimensional singular set. In fact, $\phi(x,y) = \frac{x}{|x|}$: $\mathbb{R}^3\times\mathbb{R}^{n-3}\to S^2$ is a minimizing harmonic map whose singular set is $\{0\}\times\mathbb{R}^{n-3}$. In general, Schoen and Uhlenbeck have shown in their pioneering work [171] that Hausdorff dimension of the singular set of any minimizing harmonic map is of at most $(n-3)$-dimension.
To outline the main theorem of [171], we first introduce the notion of Hausdorff dimension.

Definition 2.2.3 For $0\le s\le n$, the s-dimensional Hausdorff measure on $\mathbb{R}^n$, H^s, is defined by

$$H^s(A) = \lim_{\delta\downarrow 0^+} H^s_\delta(A), \quad \forall A\subset\mathbb{R}^n,$$

where

$$H^s_\delta(A) = \inf\left\{\sum_i r_i^s \mid A\subset\bigcup_i B_{r_i},\ r_i\le\delta\right\}.$$

The Hausdorff dimension of $A\subset\mathbb{R}^n$ is defined by

$$\dim_H(A) := \inf\left\{s : H^s(A) = 0\right\} = \sup\left\{t : H^t(A) = \infty\right\}.$$

For a map $u\in W^{1,2}(M,N)$, denote

$$\mathrm{sing}(u) = \{x\in M \mid u \text{ is discontinuous at } x\}$$

as the singular set of u.

Theorem 2.2.4 *For $n \geq 3$, let $u \in W^{1,2}(M, N)$ be a minimizing harmonic map. Then $sing(u)$ is a closed set, which is discrete for $n = 3$ and has Hausdorff dimension at most $(n-3)$ for $n \geq 4$. Moreover, $u \in C^\infty(M \setminus sing(u), N)$.*

In order to prove Theorem 2.2.4, we need several important lemmas. The first one is an energy monotonicity inequality for minimizing harmonic maps. To simplify the presentation, we assume throughout this section that $M = \Omega \subset \mathbb{R}^n$ is a bounded domain in $\mathbb{R}^n$ equiped with the standard metric.

Lemma 2.2.5 *For $n \geq 3$, if $u \in W^{1,2}(\Omega, N)$ is a minimizing harmonic map, then for any $x \in \Omega$ and $0 < r \leq R < \operatorname{dist}(x, \partial\Omega)$,*

$$R^{2-n} \int_{B_R(x)} |\nabla u|^2 - r^{2-n} \int_{B_r(x)} |\nabla u|^2$$
$$\geq \int_{B_R(x) \setminus B_r(x)} |y - x|^{2-n} \left| \frac{\partial u}{\partial |y - x|} \right|^2. \tag{2.20}$$

Proof. For simplicity, assume $x = 0$. For $r > 0$, let $u_r(x) = u(\frac{rx}{|x|})$ for $x \in B_r$. Then, by the minimality, we have

$$\begin{aligned} \int_{B_r} |\nabla u|^2 &\leq \int_{B_r} |\nabla u_r|^2 \\ &= r^2 \int_0^r t^{n-3} \int_{S^{n-1}} |\nabla_{S^{n-1}} u|^2 \, (r\theta) \, dH^{n-1}(\theta) \\ &= \frac{r}{n-2} \int_{\partial B_r} |\nabla_T u|^2 \, dH^{n-1} \\ &= \frac{r}{n-2} \int_{\partial B_r} \left(|\nabla u|^2 - |\frac{\partial u}{\partial r}|^2 \right) dH^{n-1}, \end{aligned}$$

where $\nabla_{S^{n-1}}$ and ∇_T denote the gradient on S^{n-1} and ∂B_r respectively. This implies that for a.e. $r > 0$,

$$\begin{aligned} \frac{d}{dr} \left(r^{2-n} \int_{B_r} |\nabla u|^2 \right) &= (2-n) r^{1-n} \int_{B_r} |\nabla u|^2 + r^{2-n} \int_{\partial B_r} |\nabla u|^2 \\ &= r^{1-n} \left(r \int_{\partial B_r} |\nabla u|^2 + (2-n) \int_{B_r} |\nabla u|^2 \right) \\ &\geq r^{2-n} \int_{\partial B_r} |\frac{\partial u}{\partial r}|^2 \, dH^{n-1}. \end{aligned}$$

Integrating it from r to R, we obtain (2.20). □

As a consequence, we can easily show

Corollary 2.2.6 *For $n \geq 3$, if $u \in W^{1,2}(\Omega, N)$ is a minimizing harmonic map, then*

$$\Theta^{n-2}(u, x) = \lim_{r \downarrow 0^+} r^{2-n} \int_{B_r(x)} |\nabla u|^2$$

exists and is upper semicontinuous for all $x \in \Omega$. *Moreover, if for* $x \in \Omega$

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2 = \Theta^{n-2}(u, x)$$

for some $r > 0$, *then*

$$u(y) = u\left(r\frac{y-x}{|y-x|}\right) \text{ for a.e. } y \in B_r(x).$$

We now prove an energy decay estimate under the smallness condition.

Lemma 2.2.7 *For* $n \geq 3$, *there exist* $\epsilon_0 = \epsilon_0(n, N) > 0$ *and* $\theta_0 \in (0, \frac{1}{2})$ *such that if* $u : B_1 \to N$ *is a minimizing harmonic map satisfying*

$$E(u, B_1) \leq \epsilon_0^2,$$

then

$$\theta_0^{2-n} \int_{B_{\theta_0}} |\nabla u|^2 \leq \frac{1}{2} \int_{B_1} |\nabla u|^2. \tag{2.21}$$

Proof. We present the original proof by [171]. Denote $\epsilon^2 = E(u, B_1)$. Let $\phi \in C^\infty(\mathbb{R}^n, \mathbb{R}_+)$ be a radial mollifying function so that supp $\phi \subseteq B_1$ and $\int_{\mathbb{R}^n} \phi = 1$. Let $\bar{h} = \epsilon^{\frac{1}{2}}$, $\tau = \epsilon^{\frac{1}{4}}$, and $\theta \in (\tau, \frac{1}{4})$. Define $h = h(r), r = |x|$, to be a nondecreasing smooth function satisfying

$$h(r) = \bar{h} \quad \text{for} \quad r \leq \theta, \quad h(\theta + \tau) = 0, \quad \text{and} \quad |h'(r)| \leq 2\epsilon^{\frac{1}{4}}.$$

Then set

$$u^{(h(x))}(x) = \int_{B_1} \phi^{(h(x))}(x-y)u(y)\,dy, \quad x \in B_{\frac{1}{2}},$$

where $\phi^{(h(x))} = h(x)^{-n}\phi\left(\frac{x}{h(x)}\right)$.

For $x \in B_{\frac{1}{2}}$, by a modified version of the Poincareé inequality and the monotonicity inequality (2.20), we have

$$\begin{aligned} \operatorname{dist}^2\left(u^{(h(x))}, N\right) &\leq \frac{1}{|B_{h(x)}(x)|} \int_{B_{h(x)}(x)} \left|u(y) - u^{(h(x))}\right|^2 dy \\ &\leq Ch(x)^{2-n} \int_{B_{h(x)}(x)} |\nabla u|^2\,dy \leq C\epsilon^2. \end{aligned}$$

Hence for $\epsilon_0 > 0$ sufficiently small, $u^{(h(x))}(B_{\frac{1}{2}}) \subset N_{\delta_0}$ and we can define

$$u_{h(x)}(x) = \Pi_N\left(u^{(h(x))}(x)\right) : B_{\frac{1}{2}} \to N.$$

It is readily seen that

$$\int_{B_{\frac{1}{2}}} \left|\nabla u^{(h(x))}\right|^2 dx \leq CE(u, B_1) = C\epsilon^2,$$

$$\left|\nabla u^{(h(x))}\right|^2(x) \le Ch(x)^{-n}E(u, B_{(h(x))}(x)) \le Ch(x)^{-2}\epsilon^2 \le C\epsilon \text{ for } x \in B_{\frac{1}{2}}$$

so that

$$\sup_{x\in B_{\frac{1}{2}}} \left|u^{(h(x))} - u^{(h(x))}(0)\right| \le C\epsilon^{\frac{1}{2}}.$$

Denote $u^h = u^{(h(x))}$ and $u_h = u_{h(x)}$. Then we claim

$$\int_{B_{\theta+\tau}\setminus B_\theta} |\nabla u_h|^2 \le C \int_{B_{\theta+2\tau}\setminus B_{\theta-\tau}} |\nabla u|^2 \tag{2.22}$$

It suffices to show (2.22) for u^h. Since

$$u^h(x) = \int_{B_1} \phi(y)u(x - h(x)y)\, dy,$$

we have

$$\left|\nabla u^h\right|(x) \le \int_{B_1} |\phi|(y) \left(|\nabla u| + |\nabla h|\, |y|\, |\nabla u|\right)(x - h(x)y)\ dy,$$

so that by a change of variables we obtain

$$\begin{aligned}\int_{B_{\theta+\tau}\setminus B_\theta} \left|\nabla u^h\right|^2 &\le C\int_{B_{\theta+\tau}\setminus B_\theta}\int_{B_1} \phi(y)|\nabla u|^2(x-hy)\, dydx \\ &\le C\int_{B_{\theta+2\tau}\setminus B_{\theta-\tau}} |\nabla u|^2\, dx.\end{aligned}$$

Now let v solve the Dirichlet problem

$$\begin{aligned}\Delta v &= 0 \quad \text{in} \quad B_{\frac{1}{2}} \\ v &= u_{\bar h} \quad \text{on} \quad \partial B_{\frac{1}{2}}.\end{aligned}$$

Then we have

$$\sup_{B_{\frac{1}{2}}} |v - u_{\bar h}| \le \mathrm{osc}_{B_{\frac{1}{2}}} u_{\bar h} \le C\epsilon^{\frac{1}{2}},$$

$$\sup_{B_{\frac{1}{4}}} |\nabla v|^2 \le C\int_{B_{\frac{1}{2}}} |\nabla v|^2 \le C\int_{B_{\frac{1}{2}}} |\nabla u_{\bar h}|^2 \le CE(u, B_1) \le C\epsilon^2.$$

Hence

$$\begin{aligned}\theta^{2-n}E(u_{\bar h}, B_\theta) &\le 2\theta^{2-n}\left(\int_{B_\theta} |\nabla(u_{\bar h} - v)|^2 + \int_{B_\theta} |\nabla v|^2\right) \\ &\le 2\theta^{2-n}\int_{B_{\frac{1}{2}}} |\nabla\left(u_{\bar h} - v\right)|^2 + C\theta^2\epsilon^2.\end{aligned}$$

On the other hand

$$\int_{B_{\frac{1}{2}}} |\nabla\left(u_{\bar h} - v\right)|^2 = -\int_{B_{\frac{1}{2}}} \Delta u_{\bar h} \cdot \left(u_{\bar h} - v\right),$$

and, since u is a weakly harmonic map, we can estimate

$$\begin{aligned}\Delta u_{\bar{h}}(x) &= \int_{\mathbb{R}^n} \left[\Delta_x \phi^{\bar{h}}(x-y)\right] u(y)\, dy \\ &= \int_{\mathbb{R}^n} \left[\Delta_y \phi^{\bar{h}}(x-y)\right] u(y)\, dy \\ &= \int_{\mathbb{R}^n} \phi^{\bar{h}}(x-y) A(u)\left(\nabla u, \nabla u\right)(y)\, dy,\end{aligned}$$

so that

$$\int_{B_{\frac{1}{2}}} |\Delta u_{\bar{h}}| \leq C E_1(u, B_1).$$

Therefore we obtain

$$\theta^{2-n} E(u_{\bar{h}}, B_\theta) \leq C(\theta^{2-n}\epsilon^{\frac{1}{2}} + \theta^2) E(u, B_1). \tag{2.23}$$

Let $\alpha_n \in (0, \frac{1}{8}]$ to be chosen later, and let $\bar{\theta} = \epsilon^{\alpha_n}$. Let $p = [\frac{\bar{\theta}}{3\tau}] \quad (\geq \frac{1}{3}\epsilon^{-\frac{1}{8}} - 1)$ be the integer part of $\frac{\bar{\theta}}{3\tau}$ and write

$$[\bar{\theta}, 2\bar{\theta}] = \bigcup_{1\leq i\leq p} I_i, \quad |I_i| = 3\tau,$$

where each I_i is a closed interval of length 3τ. We have

$$\int_{r\in[\bar{\theta}, 2\bar{\theta}]} |\nabla u|^2\, dx = \sum_{1\leq i\leq p} \int_{r\in I_i} |\nabla u|^2\, dx \leq E(u, B_1).$$

Thus this is an interval I_j, $1 \leq j \leq p$, such that

$$\int_{r\in I_j} |\nabla u|^2 \leq p^{-1} E(u, B_1) \leq C\epsilon^{\frac{1}{8}} E(u, B_1).$$

Let $\theta \in [\bar{\theta}, 2\bar{\theta}]$ be such that $I_j = [\theta - \tau, \theta + 2\tau]$, and let h and $u_{h(x)}$ be defined as above. Then by (2.23) we have

$$\begin{aligned}\bar{\theta}^{2-n} E\left(u, B_{\bar{\theta}}\right) &\leq 2^{n-2}\theta^{2-n} E\left(u, B_{\theta+\tau}\right) \\ &\leq 2^{n-2}\theta^{2-n} E\left(u_{h(x)}, B_{\theta+\tau}\right) \\ &= 2^{n-2}\left(\theta^{2-n} E\left(u_{\bar{h}}, B_\theta\right) + \theta^{2-n}\int_{B_{\theta+\tau}\setminus B_\theta} |\nabla u_{h(x)}|^2\right) \\ &\leq C\left(\theta^{2-n}\epsilon + \theta^2\right) E(u, B_1) + C\int_{I_j} |\nabla u|^2 \\ &\leq C\left(\theta^{2-n}\epsilon + \theta^2\right) E(u, B_1) + C\epsilon^{\frac{1}{8}} E(u, B_1) \\ &\leq C\left(\bar{\theta}^{2-n}\epsilon^{\frac{1}{2}} + \bar{\theta}^2 + \epsilon^{\frac{1}{8}}\right) E(u, B_1).\end{aligned}$$

Choosing $\alpha_n = \min\{\frac{2}{n}, \frac{1}{16}\}$, this implies

$$\bar{\theta}^{2-n}\int_{B_{\bar{\theta}}} |\nabla u|^2 \leq C\epsilon^{\frac{1}{8}} E(u, B_1).$$

This implies (2.21) provide that ϵ is chosen to be sufficiently small. □

A direct consequence of Lemma 2.2.7 is

Corollary 2.2.8 *Under the same conditions as Theorem 2.2.4, there holds*

$$sing(u) = \left\{x \in \Omega \mid \Theta^{n-2}(u, x) \geq \epsilon_0^2\right\} = \left\{x \in \Omega \mid \Theta^{n-2}(u, x) > 0\right\} \tag{2.24}$$

and $H^{n-2}(sing(u)) = 0$.

Proof. If $x_0 \notin \mathrm{sing}(u)$, then u is continuous at x_0. By an argument similar to §2.1 and the standard higher regularity theory of elliptic systems, we conclude that u is smooth near x_0 and hence $\Theta^{n-2}(u, x_0) = 0$. This implies

$$\Sigma(u) := \left\{x \in \Omega \mid \Theta^{n-2}(u, x) \geq \epsilon_0^2\right\} \subseteq \mathrm{sing}(u).$$

On the other hand, if $x_0 \notin \Sigma(u)$, then there is $r_0 > 0$ such that

$$r_0^{2-n} \int_{B_{r_0}(x_0)} |\nabla u|^2 \leq \epsilon_0^2,$$

this implies

$$(\frac{r_0}{2})^{2-n} \int_{B_{\frac{r_0}{2}}(x)} |\nabla u|^2 \leq 2^{n-2}\epsilon_0^2, \quad \forall\, x \in B_{\frac{r_0}{2}}.$$

Hence by (2.20) we have

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2 \leq 2^{n-2}\epsilon_0^2, \quad \forall x \in B_{\frac{r_0}{2}}(x_0) \text{ and } 0 < r \leq \frac{r_0}{2}.$$

By repeatedly applying Lemma 2.21, we can obtain

$$r^{2-n} \int_{B_r(x)} |\nabla u|^2 \leq C r^{2\alpha}, \quad \forall\, x \in B_{\frac{r_0}{2}}(x_0) \text{ and } 0 < r \leq \frac{r_0}{2}$$

for some $\alpha \in (0, 1)$ depending only on ϵ_0, M, N. Hence Lemma 2.1.10 implies $u \in C^\alpha(B_{\frac{r_0}{2}}(x_0), N)$ and $x_0 \notin \mathrm{sing}(u)$.

We now show $H^{n-2}(\mathrm{sing}(u)) = 0$. By (2.24), we have for any $\delta > 0$,

$$\mathcal{J} := \left\{ B_r(x) \mid x \in \mathrm{sing}(u), 0 < r < \delta \text{ s.t. } r^{2-n} \int_{B_r(x)} |\nabla u|^2 \geq \epsilon_0^2 \right\}$$

forms an open cover of $\mathrm{sing}(u)$. By Vitali's covering lemma (Evans-Gariepy [48]) there exist disjoint balls $\{B_{r_i}(x_i)\}_{i \in I}$ of $\mathcal{J}$ such that

$$\mathrm{sing}(u) \subseteq \bigcup_{i \in I} B_{5r_i}(x_i)$$

so that

$$
\begin{aligned}
H_{5\delta}^{n-2}(\mathrm{sing}(u)) &\le \sum_{i\in I}(5r_i)^{n-2} = 5^{n-2}\sum_{i\in I} r_i^{n-2} \\
&\le \frac{5^{n-2}}{\epsilon_0^2}\sum_{i\in I}\int_{B_{r_i}(x_i)} |\nabla u|^2\,dx \\
&= \frac{5^{n-2}}{\epsilon_0^2}\int_{\underset{i\in I}{\cup} B_{r_i}(x_i)} |\nabla u|^2\,dx \\
&\le C\int_{(\mathrm{sing}(u))_\delta} |\nabla u|^2 < +\infty
\end{aligned}
$$

where $(\mathrm{sing}(u))_\delta$ is δ-neighborhood of $\mathrm{sing}(u)$. Hence $H^{n-2}(\mathrm{sing}(u)) = 0$ after taking $\delta \to 0$. □

The refined dimension estimate on $\mathrm{sing}(u)$ is based on both the compactness of minimizing harmonic maps in $W^{1,2}(\Omega, N)$ and Federer's dimension reduction argument. In order to establish the compactness property of minimizing harmonic maps, we need an extension lemma, which was first proved by [171] and later by Luckhaus [139] (The readers can also consult with the lecture note by Simon [187, 188]).

Lemma 2.2.9 *For $n \ge 2$, suppose $u, v \in H^1(S^{n-1}, N)$. Then for any $\epsilon \in (0,1)$ there is $w \in H^1(S^{n-1} \times [1-\epsilon, 1], \mathbb{R}^L)$ such that $w|_{S^{n-1}\times\{1\}} = u$, $w|_{S^{n-1}\times\{1-\epsilon\}} = v$,*

$$
\begin{aligned}
\int_{S^{n-1}\times[1-\epsilon,1]} |\nabla w|^2\,dx &\le C\epsilon\int_{S^{n-1}} \left(|\nabla_T u|^2 + |\nabla_T v|^2\right) \\
&\quad + C\epsilon^{-1}\int_{S^{n-1}} |u-v|^2, \qquad (2.25)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathrm{dist}^2(w(x), N) &\le C\epsilon^{1-n}\left(\int_{S^{n-1}} \left(|\nabla_T u|^2 + |\nabla_T v|^2\right)\right)^{\frac{1}{2}} \\
&\quad \cdot\left(\int_{S^{n-1}} |u-v|^2\right)^{\frac{1}{2}} + C\epsilon^{-n}\int_{S^{n-1}} |u-v|^2 \qquad (2.26)
\end{aligned}
$$

for a.e. $x \in S^{n-1} \times [1-\epsilon, 1]$. Here ∇_T is the gradient on S^{n-1}.

We will not present the proofs of this lemma by [171] and [139], which are rather delicate. Instead, we will present a proof of an extension lemma by Hardt-Lin [83], in which N is assumed to be simply connected. More precisely, we have

Lemma 2.2.10 *If $\Omega \subset \mathbb{R}^n$ is a bounded C^2 domain and $N \subset \mathbb{R}^L$ is simply connected (i.e., $\pi_0(N) = \pi_1(N) = 0$), then any map $\eta : \partial\Omega \to N$ belonging to $H^{\frac{1}{2}}(\partial\Omega, \mathbb{R}^L)$ admits an extension $\omega \in H^1(\Omega, N)$. Moreover, for $\Omega = B \subset \mathbb{R}^n$ and $\xi \in \mathbb{R}^L$, ω may be chosen to satisfy the estimate*

$$
\int_B |\nabla\omega|^2 \le \lambda\int_{\partial B} |\nabla\eta|^2\,dH^{n-1} + c\lambda^{-1}\int_{\partial B} |\eta-\xi|^2\,dH^{n-1} \qquad (2.27)
$$

for any $0 < \lambda < 1$, where the constant c depends only on n and N.

The proof of Lemma 2.2.10 is based on the following lemma on the existence of Lipschitz retraction maps.

Lemma 2.2.11 *If $N \subset \mathbb{R}^L$ satisfies $\pi_0(N) = \pi_1(N) = 0$, then there exists a compact $(L-1)$-dimensional Lipschitz polyhedron X in $\mathbb{R}^L$ and a locally Lipschitz retraction map $P : \mathbb{R}^L \setminus X \to N$ so that*

$$\int_{B_R} |\nabla P|^2(y) < +\infty \text{ whenever } 0 < R < +\infty. \tag{2.28}$$

Proof. See [83] for the details. We sketch it here. First choose a Lipschitz nearest point projection $\Pi : \overline{N_\delta} \to N$. Thus $\pi_0(\overline{N_\delta}) = \pi_1(\overline{N_\delta}) = 0$. Let X be the interior $(L-3)$- skeleton of $S^L \setminus \overline{N_\delta}$ for some Lipschitz triangulation of $S^L \setminus N_\delta$, using the usual compactification $S^L = \mathbb{R}^L \cup \{\infty\}$ and placing the point ∞ on the interior of a L-simplex. Then there is a strong deformation retraction of $(\mathbb{R}^L \setminus X, \overline{N_\delta})$ onto $(W \cup \overline{N_\delta}, \overline{N_\delta})$ for some 2-dimensional Lipschitz complex W in $\mathbb{R}^L \setminus N_\delta$. By Alexander duality and Eilenberg extension theorem we can extend the identity map of $\overline{N_\delta}$ to $W \cup \overline{N_\delta}$ and hence $\mathbb{R}^L \setminus X$. Denote such an extension map as Φ. Since the constructions are piecewise linear, we can assume $|\nabla \Phi|(y) \leq \frac{c}{\text{dist}(y,X)}$ for a constant depending only on N. Now letting

$$P = \begin{cases} \Pi & \text{on } N_\delta \\ \Pi \circ \Phi & \text{on } \mathbb{R}^L \setminus N_\delta, \end{cases}$$

we find that

$$\int_{B_R} |\nabla P|^2 \leq [c\text{Lip}(\Pi)]^2 \int_{B_R} \text{dist}^{-2}(y, X).$$

To see that the latter integral is finite, we may assume that X is an affine space of dimension at most $(L-3)$, so that

$$\int_{B_R} \text{dist}^{-2}(y, X)$$

is finite by Fubini's theorem. □

Proof of Lemma 2.2.10:
First let $h \in C^\infty(\Omega, \mathbb{R}^L)$ be an extension of ω such that $h \in H^1 \cap L^\infty(\Omega, \mathbb{R}^L)$ (e.g. h is a harmonic extension). However, the image of h may not be contained in N. To correct this, we compose h with a suitable projection onto N.
Choose $P : \mathbb{R}^L \setminus X \to N$ as in Lemma 2.21, and let $B \subset \mathbb{R}^L$ be a large ball containing $N \cup X$. For $\tau < min\{\delta, \text{dist}(N \cup X, \partial B)\}$ and a point $a \in B_\tau = \{y \in \mathbb{R}^L \mid |y| < \tau\}$, define

$$B_a = B + \{a\} \text{ and } X_a = X + \{a\}$$

and the projection $P_a : B_a \setminus X_a \to N$ by $P_a(y) = P(y - a)$. For τ sufficiently small,

$$\Lambda = \sup_{a \in B_\tau} \text{Lip}(P_a|N)^{-1}$$

is, by the inverse function theorem, a finite number depending only on N. Since h is smooth on Ω, Sard's theorem implies that $P_a \circ h \in H^1(\Omega, N)$ for almost all $a \in B_\tau$. Using Fubini's theorem and (2.28), we infer that

$$\begin{aligned}\int_{B_\tau}\int_\Omega |\nabla(P_a \circ h)|^2 &\leq \int_\Omega |\nabla h|^2(x)\left(\int_{B_\tau} |\nabla P_a|^2 (h(x))\right) dxda\\ &\leq \int_\Omega |\nabla h|^2(x)\left(\int_{B_\tau} |(\nabla P)(h(x)-a)|^2\right) dadx\\ &\leq \int_\Omega |\nabla h|^2(x)\left(\int_B |\nabla P|^2\right) dx\\ &\leq C\int_\Omega |\nabla h|^2 < +\infty,\end{aligned}$$

where C depends on N. Thus we may choose $a \in B_\tau$ so that

$$\int_\Omega |\nabla(P_a \circ h)|^2 \leq C|B_\tau|^{-1}\int_\Omega |\nabla h|^2 .$$

We conclude that

$$\omega = (P_a|N)^{-1} \circ P_a \circ h \in H^1(\Omega, N) \text{ and } \omega|_{\partial\Omega} = \eta.$$

To prove the second conclusion we assume that $\int_{\partial\Omega} |\nabla\eta|^2 \, dH^{n-1}$ is finite and choose h to be the harmonic extension of η. Then the conclusion follows from the following lemmas.

Lemma 2.2.12 *Suppose $h \in W^{1,2}(B, \mathbb{R}^L)$ is a harmonic function. Then for any $\lambda \in (0,1)$, $\xi \in \mathbb{R}^L$, we have*

$$\int_B |\nabla h|^2 \leq C\left[\lambda \int_{\partial B} |\nabla h|^2 \, dH^{n-1} + \lambda^{-1}\int_{\partial B} |h-\xi|^2 \, dH^{n-1}\right]. \tag{2.29}$$

Proof. Define

$$v(r,\theta) = \begin{cases} \xi, & 0 \leq r \leq 1-\lambda \\ \frac{(r-(1-\lambda))}{\lambda} h(1,\theta) + \frac{1-r}{\lambda}\xi, & 1-\lambda \leq r \leq 1. \end{cases}$$

Since $v(1,\theta) = h(1,\theta)$ and h is harmonic, we have

$$\begin{aligned}\int_B |\nabla h|^2 &\leq \int_B |\nabla v|^2\\ &= \int_{B\setminus B_{1-\lambda}} \left(|v_r|^2 + \frac{1}{r^2}|\nabla_\theta v|^2\right)\\ &= \int_{B\setminus B_{1-\lambda}} \left(\frac{|h(1,\theta)-\xi|^2}{\lambda^2} + \frac{1}{r^2}(\frac{r-(1-\lambda)}{\lambda})^2|\nabla_\theta h|^2\right)\\ &\leq C\left[\lambda^{-1}\int_{\partial B} |h-\xi|^2 \, dH^{n-1} + \lambda\int_{\partial B} |\nabla_\theta h|^2 \, dH^{n-1}\right].\end{aligned}$$

This completes the proof. □

As an immediate consequence of Lemma 2.2.9, we have the following compactness result of minimizing harmonic maps, due to [171] and [140].

Lemma 2.2.13 *Let $u_i \in W^{1,2}(\Omega, N)$ be a sequence of minimizing harmonic maps. If $u_i \to u$ weakly in $W^{1,2}(\Omega, N)$, then $u_i \to u$ strongly in $W^{1,2}_{\mathrm{loc}}(\Omega, N)$ and u is a minimizing harmonic map.*

Proof. For any unit ball $B_1 \subset\subset \Omega$ and a small $\lambda \in (0,1)$, let $w \in H^1(B, N)$ be such that $w = u$ on $B_1 \setminus B_{1-\lambda}$. By Fatou's lemma and Fubini's theorem, there is $\rho \in (1-\lambda_0, 1)$ such that

$$\lim_{i\to\infty} \int_{\partial B_\rho} |u_i - u|^2 \, dH^{n-1} = 0, \quad \int_{\partial B_\rho} \left(|\nabla u_i|^2 + |\nabla w|^2\right) dH^{n-1} \le C < +\infty.$$

Applying Lemma 2.2.9 to u_i and w, we conclude that there is $v_i \in H^1(B_\rho, \mathbb{R}^L)$ such that for suitable $\lambda_i \downarrow 0$,

$$v_i(x) = \begin{cases} w\left(\frac{x}{1-\lambda_i}\right), & |x| \le \rho(1-\lambda_i) \\ u_i(x), & |x| = \rho, \end{cases}$$

$$\begin{aligned} & \int_{B_\rho \setminus B_{\rho(1-\lambda_i)}} |\nabla v_i|^2 \\ \le \; & C\left[\lambda_i \int_{\partial B_\rho} \left(|\nabla u_i|^2 + |\nabla w|^2\right) dH^{n-1} + \lambda_i^{-1} \int_{\partial B_\rho} |u_i - u|^2 \, dH^{n-1}\right] \\ & \to 0 \ \text{ as } \ i \to \infty, \end{aligned} \tag{2.30}$$

and $\mathrm{dist}(v_i, X) \to 0$ uniformly in $B_\rho \setminus B_{\rho(1-\lambda_i)}$ as $i \to \infty$. Define a comparison map for u_i by

$$w_i = \begin{cases} w\left(\frac{x}{1-\lambda_i}\right), & |x| \le \rho(1-\lambda_i) \\ \Pi_N\left(v_i(x)\right), & \rho(1-\lambda_i) \le |x| \le \rho. \end{cases}$$

Then by minimality and (2.30) we have

$$\begin{aligned} \int_{B_\rho} |\nabla u|^2 &\le \lim_{i\to\infty} \int_{B_\rho} |\nabla u_i|^2 \le \lim_{i\to\infty} \int_{B_\rho} |\nabla w_i|^2 \\ &= \lim_{i\to\infty} \left[\int_{B_{\rho(1-\lambda_i)}} \left|\nabla w\left(\frac{\cdot}{1-\lambda_i}\right)\right|^2 + \int_{B_\rho \setminus B_{\rho(1-\lambda_i)}} \left|\nabla\left(\Pi_N(v_i)\right)\right|^2\right] \\ &\le \lim_{i\to\infty} \left[(1-\lambda_i)^{n-2} \int_{B_\rho} |\nabla w|^2 + C\mathrm{Lip}(\Pi_N)^2 \int_{B_\rho \setminus B_{\rho(1-\lambda_i)}} |\nabla v_i|^2\right] \\ &\le \int_{B_\rho} |\nabla w|^2. \end{aligned}$$

This implies both minimality of u and strong convergence of u_i to u. □

Remark 2.2.14 (1) The extension Lemma 2.2.10 may fail if N is not simply connected. For example, let $\Omega = B \subset \mathbb{R}^3$, $N = S^1 \subset \mathbb{R}^2$. Then

$$\eta(x_1, x_2, x_3) = \frac{(x_1, x_2)}{|(x_1, x_2)|} \text{ for } (x_1, x_2, x_3) \in \partial B \setminus \{(0,0,1), (0,0,-1)\}$$

does not have an extension in $H^1(B, S^1)$.
(2) If $N \subset \mathbb{R}^L$ is simply connected, then for any $\tilde{\Omega} \subset\subset \Omega$ there exists $C(\tilde{\Omega}, \Omega, N)$ depending only on $\tilde{\Omega}, \Omega$ and N such that any minimizing harmonic map $u \in H^1(\Omega, N)$ satisfies the uniform energy estimate

$$\int_{\tilde{\Omega}} |\nabla u|^2 \leq C(\tilde{\Omega}, \Omega, N) \tag{2.31}$$

(3) (2.31) may fail if N is not simply connected.

Proof. (1) By constructing an extension of η in $H^1(B, \mathbb{R}^2)$ by giving u two point singularities, one at $(0,0,1)$ and one at $(0,0,-1)$, we see that $\eta \in H^{\frac{1}{2}}(\partial B, S^1)$. On the other hand, there exists no extension of η in $H^1(\Omega, S^1)$. For, otherwise, there would exist an extension $u \in H^1(B, S^1)$ of minimal energy. Then, by the interior and boundary regualarity theorey to be presented below, the restriction to a horizontal disk, $u|\bar{B} \cap \{x_3 = a\}$, would be continuous for all but finitely many $a \in (-1,1)$. But this is impossible because $u|_{\partial(\bar{B} \cap \{x_3 = a\})}$ is a homeomorphism.
(2) First we claim that for any $\lambda \in (0,1)$ and any ball $B_{2R} \subset \Omega$, it holds

$$R^{2-n} \int_{B_R} |\nabla u|^2 \leq C \left[\lambda (2R)^{2-n} \int_{B_{2R}} |\nabla u|^2 + (2R)^{-n} \int_{B_{2R}} |u - u_{2R}|^2 \right], \tag{2.32}$$

where u_{2R} is the average of u over B_{2R}.

In fact, by Fubini's theorem there exists $R_1 \in (R, 2R)$ such that

$$R_1^{3-n} \int_{\partial B_{R_1}} |\nabla u|^2 \leq 2^n (2R)^{2-n} \int_{B_{2R}} |\nabla u|^2,$$

$$R_1^{1-n} \int_{\partial B_{R_1}} |u - u_{2R}|^2 \, dH^{n-1} \leq 2^{n+1} (2R)^{-n} \int_{B_{2R}} |u - u_{2R}|^2.$$

Applying Lemma 2.2.10 with $\Omega = B_{R_1}$ and $\xi = u_{2R}$, there is an extension $w \in H^1(B_{R_1}, N)$ of $u|_{\partial B_{R_1}}$ such that

$$\begin{aligned} & R_1^{2-n} \int_{B_{R_1}} |\nabla w|^2 \\ & \leq C \left(\lambda R_1^{3-n} \int_{\partial B_{R_1}} |\nabla u|^2 + \lambda^{-1} R_1^{1-n} \int_{\partial B_{R_1}} |u - u_{2R}|^2 \, dH^{n-1} \right) \\ & \leq C \left(\lambda (2R)^{2-n} \int_{B_{2R}} |\nabla u|^2 + \lambda^{-1} (2R)^{-n} \int_{B_{2R}} |u - u_{2R}|^2 \right). \end{aligned}$$

This, combined with the fact that $\int_{B_{R_1}} |\nabla u|^2 \leq \int_{B_{R_1}} |\nabla w|^2$ (since u is energy minimizing), implies (2.32). It is well-known (see Giaquinta [65]) that iterations of (2.32) implies

$$R^{2-n} \int_{B_R} |\nabla u|^2 \leq C(2R)^{-n} \int_{B_{2R}} |u - u_{2R}|^2 \leq C(N) \tag{2.33}$$

whenever $B_{2R} \subset \Omega$. For any subdomain $\tilde{\Omega} \subset\subset \Omega$, we can apply simple covering argument and (2.33) to obtain (2.31).
(3) Note that for $j \geq 1$, $u_j(x_1, \cdots, x_n) = (\cos jx_1, \sin jx_1) : B \subset \mathbb{R}^n \to S^1 \subset \mathbb{R}^2$ are minimizing harmonic maps that have unbounded energy on each subdomain. □

2.3 Federer's dimension reduction principle

In this section we will first present the dimension reduction principle, which was first developed by Federer [54] in the study of area minimizing currents, of minimizing harmonic maps by [171]. Then we will complete the proof of Theorem 2.2.4. Here we follow the presentation by Simon [187, 188].
We begin with

Definition 2.3.1 Let $u \in H^1(\Omega, N)$ be a minimizing harmonic map and $x_0 \in \text{sing}(u)$, a map $\phi \in H^1_{\text{loc}}(\mathbb{R}^n, N)$ is a *tangent map* of u at x_0 if there exists $r_i \downarrow 0$ such that $u_{x_0,r_i}(x) \equiv u(x_0 + r_i x) \rightharpoonup \phi$ weakly in $H^1_{\text{loc}}(\mathbb{R}^n, N)$.

We now collect some basic properties of tangent maps.

Proposition 2.3.2 *Let $u \in H^1(\Omega, N)$ be a minimizing harmonic map and $\phi \in H^1_{\text{loc}}(\mathbb{R}^n, N)$ be a tangent map of u at $x_0 \in sing(u)$. Then*
(1) there exists $r_i \downarrow 0$ such that $u_{x_0,r_i} \to \phi$ in $H^1_{\text{loc}}(\mathbb{R}^n, N)$ and ϕ is a minimizing harmonic map.
(2) ϕ is homogeneous of degree zero and

$$0 < \Theta^{n-2}(u, x_0) = \Theta^{n-2}(\phi, 0) (= \rho^{2-n} \int_{B_\rho} |\nabla \phi|^2 \ \forall \rho > 0).$$

(3)

$$\Theta^{n-2}(\phi, 0) = \max_{x \in \mathbb{R}^n} \Theta^{n-2}(\phi, x). \tag{2.34}$$

(4)

$$S(\phi) \equiv \left\{ y \in \mathbb{R}^n \mid \Theta^{n-2}(\phi, y) = \Theta^{n-2}(\phi, 0) \right\} \tag{2.35}$$

is a linear subspace in $\mathbb{R}^n$. ϕ is invariant under composition with translation by elements of $S(\phi)$.

Proof. (1) directly follows from Lemma 2.2.13. (2) follows from (1) and the monotonicity formula (2.20).
For (3), note that the monotonicity formula (2.20) implies that for any $R > 0$ and $y \in \mathbb{R}^n$,

$$R^{2-n} \int_{B_R(y)} |\nabla \phi|^2 = \Theta^{n-2}(\phi, y) + 2 \int_{B_R(y)} r_y^{2-n} |\frac{\partial \phi}{\partial r_y}|^2,$$

where $r_y(x) = |x-y|$ and $\frac{\partial}{\partial r_y} = \frac{(x-y)\cdot\nabla}{|x-y|}$. Now since $B_R(y) \subset B_{R+|y|}(0)$, we have

$$\begin{aligned} R^{2-n}\int_{B_R(y)} |\nabla\phi|^2 &\le \left(1+\frac{|y|}{R}\right)^{n-2}\left((R+|y|)^{2-n}\int_{B_{R+|y|}(0)} |\nabla\phi|^2\right) \\ &= \left(1+\frac{|y|}{R}\right)^{n-2}\Theta^{n-2}(\phi,0). \end{aligned}$$

Thus letting $R\uparrow\infty$, we get

$$\Theta^{n-2}(\phi,0) \ge \Theta^{n-2}(\phi,y) + 2\int_{\mathbb{R}^n} r_y^{2-n}|\frac{\partial\phi}{\partial r_y}|^2, \tag{2.36}$$

which implies (2.34) and (3). We can also see that equality holds in (2.36) only if $\frac{\partial\phi}{\partial r_y} = 0$ a.e., that is only if $\phi(y+\lambda x) = \phi(y+x)$ for any $\lambda > 0$ and $x \in \mathbb{R}^n$. Since ϕ is also homogeneous of degree zero, we have that for any $x \in \mathbb{R}^n$ and $\lambda > 0$,

$$\begin{aligned} \phi(x) &= \phi(\lambda x) = \phi(y + (\lambda x - y)) = \phi(y + \lambda^{-2}(\lambda x - y)) \\ &= \phi(\lambda(y + \lambda^{-2}(\lambda x - y))) = \phi(x + ty) \end{aligned}$$

for any $y \in S(\phi)$, where $t = \lambda - \lambda^{-1}$ is an arbitrary real number. In particular, ϕ is a function that is independent of y-direction for any $y \in S(\phi)$. Combining this with the fact that if $z \in \mathbb{R}^n$ and $\phi(x+z) = \phi(x)$ for all $x \in \mathbb{R}^n$, then $\Theta^{n-2}(\phi,z) = \Theta^{n-2}(\phi,0)$, we conclude that $S(\phi)$ is a linear subspace and (4). □

Since 0 is a singular point for a (minimizing) tangent map ϕ, we have $S(\phi)$ is at most a $(n-1)$-dimensional subspace of $\mathbb{R}^n$. Moreover, since $\Theta^{n-2}(\phi,z) = \Theta^{n-2}(\phi,0) > 0$ for all $z \in S(\phi)$, $S(\phi)$ is a subset of sing(ϕ), the singular set of ϕ. Since it is proved in Corollary 2.2.8 that $H^{n-2}(\text{sing}(\phi)) = 0$, this yields that $\dim(S(\phi)) \le n-3$ for any tangent map ϕ.

Now for $0 \le j \le n-3$ we define a stratification of sing(u) as follows.

$$\mathcal{S}_j(u) = \{y \in \text{sing}(u) \mid \dim(S(\phi)) \le j \ \text{ for all tangent maps } \ \phi \text{ of } u \text{ at } y\}.$$

Then we have

$$\mathcal{S}_0(u) \subset \mathcal{S}_1(u) \subset \cdots \subset \mathcal{S}_{n-3}(u) = \text{sing}(u). \tag{2.37}$$

We now have a refined version, due to Almgren [4], of the dimension reduction argument of Federer [54].

Lemma 2.3.3 *For $j = 0, \cdots, n-3$, $dim_H(\mathcal{S}_j(u)) \le j$.*

As an immediate consequence, we can now complete the proof of Theorem 2.2.4.

Corollary 2.3.4 *For $n \ge 3$, let $u \in H^1(\Omega, N)$ be a minimizing harmonic map. Then sing(u) has Hausdorff dimension at most $(n-3)$, and is discrete for $n = 3$.*

Proof. The first conclusion follows directly from Lemma 2.3.3. For the second conclusion, we argue by contradiction. Suppose it were false. Then there exist $\{x_i\}$ and x_0 in $\text{sing}(u)$ such that $x_i \to x_0$. Set $r_i = |x_i - x_0| \to 0$. Define $v_i(x) = u(x_0 + r_i x) : B_2 \to N$. Then we have

$$\lim_{i\to\infty} \int_{B_2} |\nabla v_i|^2 = \lim_{i\to\infty} \left(r_i^{2-n} \int_{B_{2r_i}(x_0)} |\nabla u|^2 \right) = 2^{n-2}\Theta^{n-2}(u, x_0).$$

Hence by lemma 2.2.13 we may assume that $v_i \to v$ in $H^1_{\text{loc}}(B_2, N)$. By the monotonicity formula (2.20), we have $v(x) = v\left(\frac{x}{|x|}\right)$ a.e. in B_2. Moreover, if

$$q = \lim_{i\to\infty} \frac{x_i - x_0}{|x_i - x_0|} \in S^{n-1}$$

then both 0 and q are singular points of v and $H^1(\text{sing}(v)) > 0$, which is impossible. □

The following fact plays an important role in the proof of Lemma 2.3.3.

Lemma 2.3.5 *For each $y \in \mathcal{S}_j(u)$ and each $\delta > 0$, there is an $\epsilon > 0$ depending on u, y, δ such that for each $\rho \in (0, \delta]$*

$$\mathcal{D}_{y,\rho}\left(\{x \in B_\rho(y) \mid \Theta^{n-2}(u, x) \geq \Theta^{n-2}(u, y) - \epsilon\}\right) \subset \textit{the } \delta \textit{ neighborhood of } L_{y,\rho}$$

for some j-dimensional subspace $L_{y,\rho}$ of $\mathbb{R}^n$, where

$$\mathcal{D}_{y,\rho}(x) = \rho^{-1}(x - y), \ \forall x \in \mathbb{R}^n.$$

Proof. If it were false. Then there are $\delta > 0$ and $y \in \mathcal{S}_j(u)$ and $\rho_k \downarrow 0$, $\epsilon_k \downarrow 0$ such that

$$\{x \in B_1(0) \mid \Theta^{n-2}(u_{y,\rho_k}, x) \geq \Theta^{n-2}(u, y) - \epsilon_k\} \not\subset \text{the } \delta \text{ neighborhood of } L \quad (2.38)$$

for any j-dimensional subspace L of $\mathbb{R}^n$. But $u_{y,\rho_k} \to \phi$, a tangent map of u at y. Since $y \in \mathcal{S}_j(u)$, we have $\dim(S(\phi)) \leq j$ so there is a j-dimensional subspace $L_0 \subset S(\phi)$ and $\alpha > 0$ such that

$$\Theta^{n-2}(\phi, x) < \Theta^{n-2}(\phi, 0) - \alpha \text{ for all } x \in B_1(0) \text{ with } \text{dist}(x, L_0) \geq \delta. \quad (2.39)$$

Then by the upper semicontinuity of $\Theta^{n-2}(\cdot,\cdot)$ with respect to both arguments, we must have, for all k sufficiently large, that

$$\Theta^{n-2}(u_{y,\rho_k}, x) < \Theta^{n-2}(\phi, 0) - \alpha \text{ for all } x \in B_1(0) \text{ with } \text{dist}(x, L_0) \geq \delta. \quad (2.40)$$

But this precisely implies that for all k sufficiently large,

$$\{x \in B_1(0) \mid \Theta^{n-2}(u_{y,\rho_k}, x) \geq \Theta^{n-2}(\phi, 0) - \alpha\}$$

is contained in the δ-neighborhood of L_0, which contradicts (2.38). □

Proof of Lemma 2.3.3:
First we decompose $\mathcal{S}_j(u)$ into subsets $\mathcal{S}_{j,i}$, $i \in \{1, 2, \cdots\}$, defined to be the set of $y \in \mathcal{S}_j(u)$ such that the conclusion of Lemma 2.3.5 holds with $\epsilon = i^{-1}$. Then we have $\mathcal{S}_j(u) = \cup_{i\geq 1}\mathcal{S}_{j,i}$. Next for any integer $q \geq 1$ we let

$$\mathcal{S}_{j,i,q} = \left\{ x \in \mathcal{S}_{j,i} : \Theta^{n-2}(u,x) \in (\frac{q-1}{i}, \frac{q}{i}] \right\}$$

so that $\mathcal{S}_j(u) = \cup_{i,q}\mathcal{S}_{j,i,q}$. For $x \in \mathcal{S}_{j,i,q}$, we have that

$$\mathcal{S}_{j,i,q} \subset \left\{ x \mid \Theta^{n-2}(u,x) \geq \Theta^{n-2}(u,y) - i^{-1} \right\}$$

and hence by Lemma 2.3.5, for each $\rho \leq i^{-1}$

$$\eta_{y,\rho}\left(\mathcal{S}_{j,i,q} \cap B_\rho(y)\right) \subset \text{the } \delta \text{ neighborhood of } L_{y,\rho} \tag{2.41}$$

for some j-dimensional subspace $L_{y,\rho}$ of $\mathbb{R}^n$.
We now recall that if L is a j-dimensional subspace of $\mathbb{R}^n$ and for each $\delta \in (0, \frac{1}{8})$ we can find $\beta = \beta(\delta)$ with $\lim_{\delta\to 0}\beta(\delta) = 0$ and $\sigma = \sigma(\delta) \in (0,1)$ such that for each $R > 0$ a $2\delta R$-neighborhood of $L \cap B_R(0)$ can be covered by balls $B_{\delta R}(y_k)$ with centers $y_k \in L \cap B_R(0)$, $k = 1, \cdots, Q$ such that $Q(\delta R)^{j+\beta(\delta)} < \frac{1}{2}R^{j+\beta(\delta)}$.
Now the lemma follows by using successively finer covers of A by balls. For simplicity assume A is bounded, we first take an initial cover of A by balls $B_{\frac{\rho_0}{2}}(y_k)$ with $A \cap B_{\frac{\rho_0}{2}}(y_k) \neq \emptyset$, $k = 1, \cdots, Q$, and let $T_0 = \frac{1}{2}(\frac{\rho_0}{2})^{j+\beta(\delta)}$. For each k pick $z_k \in A \cap B_{\frac{\rho_0}{2}}(y_k)$. Then by (2.41) with $\rho = \rho_0$ there is a j-dimensional affine space L_k such that $A \cap B_{\rho_0}(z_k)$ is contained in the $\delta\rho_0$-neighborhood of L_k. Note that $L_k \cap B_{\frac{\rho_0}{2}}(y_k)$ is a j-dimensional disk of radius $\leq \frac{\rho_0}{2}$, and so we can cover its $\delta\rho_0$-neighborhood by balls $B_{\frac{\delta\rho_0}{2}}(z_{j,l})$, $l = 1, \cdots, P$ such that $P(\frac{\delta\rho_0}{2})^{j+\beta(\delta)} \leq \frac{1}{2}(\frac{\rho_0}{2})^{j+\beta(\delta)}$. Thus A can be covered by balls $B_{\frac{\delta\rho_0}{2}}(w_l)$, $l = 1, \cdots, M$, so that $M(\frac{\delta\rho_0}{2})^{j+\beta(\delta)} \leq \frac{1}{2}T_0$. Iterating q-times we can find a cover of A by balls $B_{\frac{\delta^q\rho_0}{2}}(w_k)$, $k = 1, \cdots, R_q$, so that $R_q(\frac{\delta^q\rho_0}{2})^{j+\beta(\delta)} \leq 2^{-q}T_0$. Therefore we conclude that $H^{j+\beta(\delta)}(A) = 0$ and hence A has Hausdorff dimension at most j. □

It is not hard to see from this lemma that we have a useful criterion for showing that the singular set is smaller (or empty) for minimizing maps into certain target manifolds. More precisely,

Remark 2.3.6 Suppose $3 \leq l \leq n$ is the largest integer such that there doesn't exist any nontrivial weakly harmonic map $\phi \in H^1(S^{j-1}, N)$ whose homogeneous of degree zero extension $\bar{\phi}(x) = \phi\left(\frac{x}{|x|}\right) : \mathbb{R}^j \to N$ (for $j = 3, \cdots, l$) is a minimizing harmonic map. Then any minimizing harmonic map $u \in H^1(\Omega, N)$ has its singular set $\mathcal{S}(u)$ of Hausdorff dimension at most $n - l - 1$.

Proof. By (2.37) and Lemma 2.3.3, it suffices to show that $\mathcal{S}_j(u) = \mathcal{S}_{n-l-1}(u)$ for any $n - l \leq j \leq n - 3$. For simplicity, let us just verify this for $j = n - l$. Suppose that there is a $x_0 \in \mathcal{S}_{n-l}(u) \setminus \mathcal{S}_{n-l-1}(u)$. Then by the definition we have that there is a tangent map ϕ of u at x_0 such that $\dim(S(\phi)) = n - l$. Assume $S(\phi) = \mathbb{R}^{n-l}$ and write the coordinate $z = (x, y) \in \mathbb{R}^n = \mathbb{R}^{n-l} \times \mathbb{R}^l$ for $x \in \mathbb{R}^{n-l}$ and $y \in \mathbb{R}^l$. Then $\hat{\phi}(y) \equiv \phi(z)(= \phi(\frac{y}{|y|})) : \mathbb{R}^l \to N$ gives a nontrivial minimizing harmonic map of homogeneous of degree zero, contradicting our assumption. □

Using this criterion and some Liouville theorem on stable harmonic maps into spheres, Schoen-Uhlenbeck [174] proved the following theorem, which has been extended to the class stable-stationary harmonic maps by Hong-Wang [98] and Lin-Wang [132] very recently (see §3.4 below for details).

Theorem 2.3.7 *For $k \geq 2$, if $u \in H^1(\Omega, S^k)$ is a minimizing harmonic map then $dim_H(sing(u)) \leq n - d(k) - 1$, where*

$$d(k) = \begin{cases} 2 & \text{for } k = 2 \\ 3 & \text{for } k = 3 \\ \min\{[\frac{k}{2}] + 1, 6\} & \text{for } k \geq 4 \end{cases}$$

with $[t]$ denotes the greatest integer part of t.

2.4 Boundary regularity for minimizing harmonic maps

In this section, we will discuss the boundary regularity for minimizing harmonic maps under the Dirichlet boundary value problem. In contrast with the interior regularity problem, it has been proved by Schoen-Uhlenbeck [172] that for smooth boundary data any minimizing harmonic map is completely smooth near the boundary. More precisely,

Theorem 2.4.1 *Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain and $\phi \in C^\infty(\partial\Omega, N)$ be given. Suppose $u \in H^1(\Omega, N)$ is a minimizing harmonic map with $u = \phi$ on $\partial\Omega$. Then there exists $\delta > 0$ depending on $\partial\Omega$, N, ϕ and u such that $u \in C^\infty(\Omega_\delta, N)$, where $\Omega_\delta = \{x \in \overline{\Omega} \mid \operatorname{dist}(x, \partial\Omega) \leq \delta\}$.*

The approach is similar to [171]. There are four steps: (i) boundary ϵ_0-regularity theorem, (ii) boundary compactness theorem, (iii) boundary monotonicity inequality, and (iv) the non-existence of tangent maps at any boundary point.
Since the first two steps can be proved by suitable modifications of Lemma 2.2.7 and lemma 2.2.13, we will state them without proof and indicate some details of proofs for (iii) and (iv). We start with the derivation of boundary monotonicity inequality.

Lemma 2.4.2 *For $n \geq 3$, $\Omega \subset \mathbb{R}^n$ a bounded C^1 domain and $\phi \in \operatorname{Lip}(\Omega_{\delta_0}, N)$ for some $\delta_0 > 0$, suppose $u \in H^1(\Omega, N)$ is a minimizing harmonic map with $u = \phi$ on $\partial\Omega$. Then there are $R_0 \in (0, \delta_0)$ and $C_0 > 0$ depending on $\partial\Omega$, ϕ, and N such that for any $0 < r \leq R \leq R_0$ and $x_0 \in \partial\Omega$,*

$$r^{2-n} \int_{B_r(x_0)\cap\Omega} |\nabla u|^2 + \int_{(B_R(x_0)\setminus B_r(x_0))\cap\Omega} |y - x_0|^{2-n} \left|\frac{\partial u}{\partial|y - x_0|}\right|^2$$
$$\leq e^{C_0 R} R^{2-n} \int_{B_R(x_0)\cap\Omega} |\nabla u|^2 + C_0(R - r). \tag{2.42}$$

Proof. By translation and locally flatting the boundary, we may assume $x_0 = 0$ and $B_{\delta_0}(0) \cap \Omega = B_r^+(0) \equiv B_\delta(0) \cap \{x_n \geq 0\}$. For $0 < R < \delta_0$, define $v : B_R^+(0) \to \mathbb{R}^L$ be letting

$$v(x) = \phi(x) + (u - \phi)(\frac{Rx}{|x|}) \text{ for } x \in B_R^+(0),$$

then it is easy to check that $v = \phi$ on $\partial B_R^+(0) \cap \{x_n = 0\}$ and $v = u$ on $\partial B_R^+(0) \cap \{x_n > 0\}$. Moreover

$$\text{dist}(v(x), N) \leq C\|\phi\|_{\text{Lip}} R \leq CR \text{ for } \ x \in B_R^+(0).$$

Therefore for $R_0 > 0$ sufficiently small, v lies in the δ-neighborhood of N, where $\Pi_N : N_\delta \to N$ is the smooth nearest point projection, and $\hat{u} = \Pi_N \circ v$ is a comparison map for u. Hence by the minimality we have

$$\begin{aligned}
\int_{B_R^+(0)} |\nabla u|^2 &\leq \int_{B_R^+(0)} |\nabla \hat{u}|^2 \\
&\leq \left(1 + C\|(\nabla \Pi_N)(v(x))\|_{L^2(B_R^+(0))}\right) \int_{B_R^+(0)} |\nabla v|^2 \\
&\leq (1 + C\Lambda R) \int_{B_R^+(0)} \left|\nabla \left(u(\frac{Rx}{|x|})\right)\right|^2 + C\Lambda R^{n-1},
\end{aligned}$$

where $\Lambda = \|\nabla \Pi_N\|_{L^\infty(N_\delta)}$. By direct calculations, we have

$$\begin{aligned}
\int_{B_R^+(0)} |\nabla(u(\frac{Rx}{|x|}))|^2 &= \frac{1}{n-2} R \frac{d}{dR} \left(\int_{B_R^+(0)} |\nabla u|^2 \right) \\
&\quad - \frac{1}{n-2} \int_{\partial B_R^+(0) \cap \{x_n > 0\}} |\frac{\partial u}{\partial r}|^2 \, dH^{n-1}. \qquad (2.43)
\end{aligned}$$

Putting these two inequalities together and integrating R yields (2.42). □

Next we show that there is no minimizing tangent maps at the boundary.

Theorem 2.4.3 *Any minimizing harmonic map $u_0 \in H^1(B^+, N)$ that is homogeneous of degree zero and that is constant on $B \cap \{x_n = 0\}$ must be constant.*

Proof. We follow the construction by Hardt-Lin [83]. We will consider the energy of a comparison map obtained by homogeneous of degree zero extension from a point $(0, \cdots, 0, \alpha)$, where $0 < \alpha < 1$. We use spherical polar coordinates to represent $x \in \partial B \cap \{x_n \geq 0\}$ by $(\omega, \phi) \in S^{n-2} \times [0, \frac{\pi}{2}]$.
Let θ be the angle of S^{n-1} with $(0, \cdots, 0, \alpha)$. Then we have

$$\theta = \phi + \sin^{-1}(\alpha \sin \theta).$$

As the angle ϕ varies from 0 to $\frac{\pi}{2}$, the angle θ varies from 0 to

$$\Theta(\alpha) = \pi - \sin^{-1}\left((1 + \alpha^2)^{-\frac{1}{2}}\right).$$

The distance between x and $(0,\cdots,0,\alpha)$ is $R(\phi,\alpha)=\left[(\alpha-\cos\phi)^2+\sin^2\alpha\right]^{\frac{1}{2}}$. The comparison map is given by

$$v_\alpha(\theta,\omega)=u_0(\phi,\omega) \text{ for } \theta\in[0,\Theta] \text{ and } \omega\in S^{n-2},$$

its energy $E(\alpha)$ equals

$$\begin{aligned}&\int_0^{\Theta(\alpha)}\int_0^{R(\phi,\alpha)}\int_{S^{n-2}}\left(|\frac{\partial v_\alpha}{\partial\omega}|^2\sin^{-2}\theta+|\frac{\partial v_\alpha}{\partial\theta}|^2\right)\sin^{n-2}\theta r^{n-3}\,d\omega dr d\theta\\ &=(n-2)^{-1}\int_0^{\Theta(\alpha)}\int_{S^{n-2}}\left(|\frac{\partial v_\alpha}{\partial\omega}|^2\sin^{-2}\theta+|\frac{\partial v_\alpha}{\partial\theta}|^2\right)\sin^{n-2}\theta R^{n-2}(\phi,\alpha)\,d\omega d\theta.\end{aligned}$$

Changing variables according to $\theta=\theta(\phi,\alpha):[0,\frac{\pi}{2}]\times[0,1)\to[0,\Theta(\alpha))$, $E(\alpha)$ equals

$$\begin{aligned}&(n-2)^{-1}\int_0^{\frac{\pi}{2}}\int_{S^{n-2}}\left(|\frac{\partial u_0}{\partial\omega}|^2\sin^{-2}\theta(\phi,\alpha)+|\frac{\partial u_0}{\partial\phi}|^2|\frac{\partial\phi}{\partial\theta}|^2\right)\\ &\cdot\sin^{n-2}\theta(\phi,\alpha)R^{n-2}(\phi,\alpha)|\frac{\partial\theta}{\partial\phi}|\,d\omega d\phi.\end{aligned}$$

Since $E(\alpha)$ has a minimum at $\alpha=0$, we have $I'(0+)\geq 0$. To compute it, we use the following facts:

$$R(\phi,\alpha)|_{\alpha=0}=1,\quad \frac{\partial\phi}{\partial\alpha}|_{\alpha=0}=\sin\phi,$$

$$\frac{\partial\theta}{\partial\phi}|_{\alpha=0}=1=\frac{\partial\phi}{\partial\theta}|_{\alpha=0},\quad \frac{\partial R}{\partial\alpha}|_{\alpha=0}=-\cos\phi,$$

$$\frac{\partial^2\theta}{\partial\phi\partial\alpha}|_{\alpha=0}=\cos\phi,\quad \frac{\partial^2\phi}{\partial\theta\partial\alpha}|_{\alpha=1}=-\cos\phi.$$

Letting $e(u_0)=\left(|\frac{\partial u_0}{\partial\omega}|^2\sin^{-2}\phi+|\frac{\partial u_0}{\partial\phi}|^2\right)\sin^{n-2}\phi$, we conclude that

$$\begin{aligned}0\leq(n-2)I'(0+) &= (n-2)\int_0^{\frac{\pi}{2}}\cos\phi\int_{S^{n-2}}e(u_0)\,d\omega d\phi\\ &\quad -(n-2)\int_0^{\frac{\pi}{2}}\cos\phi\int_{S^{n-2}}e(u_0)\,d\omega d\phi\\ &\quad +\int_0^{\frac{\pi}{2}}\cos\phi\int_{S^{n-2}}e(u_0)\,d\omega d\phi\\ &\quad -2\int_0^{\frac{\pi}{2}}\cos\phi\int_{S^{n-2}}e(u_0)\,d\omega d\phi\\ &= -\int_0^{\frac{\pi}{2}}\cos\phi\int_{S^{n-2}}e(u_0)\,d\omega d\phi\leq 0,\end{aligned}$$

hence $e(u_0)=0$ for almost all (ω,ϕ) and u_0 is a constant. □

We now sketch briefly how to prove Theorem 2.4.1. First note that since $\partial\Omega\in C^1$ implies that there are $R_0, C_0>0$ depending only on $\partial\Omega$ such that for any $a\in\partial\Omega$, there is $\Psi_a : \mathbb{R}^{n-1} \to \mathbb{R}$ with $\Psi(0) = 0 = |\nabla\Psi(0)|$ and $\mathrm{Lip}(\Psi_a) \le C_0$ so that after suitable rotation and translation of the coordinate system we have

$$\Omega \cap B_{R_0}(a) = \left\{x = (x_1', x_n) \in B_{R_0}^n : x_n < \Psi_a(x')\right\} (\equiv \Omega_{\Psi_a}),$$

and

$$\partial\Omega \cap B_{R_0}(a) = \left\{x = (x_1', x_n) \in B_{R_0}^n : x_n = \Psi(x')\right\} (\equiv \partial\Omega_{\Psi_a} \cap B_{R_0}).$$

Then it suffices to prove the boundary regularity for such a graphic domain.

Given a minimizing harmonic map u on Ω_Ψ with $u|_{\partial\Omega_\Psi \cap B_{R_0}} = \phi(x', \Psi(x'))$, we define an odd extension $\tilde{u}$ of u to B_{R_0} as follows.

$$\tilde{u}(x) = \begin{cases} u(x) - \phi\left(x', \Psi(x')\right) & \text{for } x \in \Omega_\Psi, \\ -\left(u(x', -x_n + 2\Psi(x')) - \phi(x', \Psi(x'))\right) & \text{for } x \in B_{R_0} \setminus \Omega_\Psi. \end{cases}$$

Note that

$$\begin{aligned} &\left| r^{2-n} \int_{B_r} |\nabla \tilde{u}|^2 - 2r^{2-n} \int_{\Omega_\Psi \cap B_r} |\nabla u|^2 \right| \\ &\le C \left(\mathrm{Lip}\phi + \mathrm{Lip}(\Psi)\right) r^{2-n} \int_{\Omega_\Psi \cap B_r} |\nabla u|^2. \end{aligned}$$

For $\tilde{u}$, we have the following ϵ_0-energy improvement lemma, which can be proved by modifying the argument of Lemma 2.2.7 (see also [83]).

Lemma 2.4.4 *There are positive constants ϵ, ν, and $\theta < 1$ such that if Ψ is as in above and $u \in H^1(\Omega_\Psi, N)$ is a minimizing harmonic map, with $u|_{B \cap \partial\Omega_\Psi} = \phi$, and $\int_{B_1} |\nabla \tilde{u}|^2 \le \epsilon^2$, then*

$$\theta^{2-n} \int_{B_\theta} |\nabla \tilde{u}|^2 \le \frac{1}{2} \max\left\{ \int_{B_1} |\nabla \tilde{u}|^2, \nu \left(Lip\ \phi + Lip\ \Psi\right) \right\}, \tag{2.44}$$

By iterating this lemma, we then get the following consequence.

Corollary 2.4.5 *Suppose that $\Omega \in C^1$, $b \in \partial\Omega$, $\phi : \partial\Omega \to N$ is C^1 near b. If $R > 0$ is small enough and $u \in H^1(\Omega, N)$ is a minimizing harmonic map with $u|_{\partial\Omega} = \phi$ and $R^{2-n} \int_{B_{2R}(b) \cap \Omega} |\nabla u|^2 \le \epsilon^2$ then $u \in C^\alpha(\overline{\Omega \cap B_R(b)}, N)$ for some $\alpha \in (0, 1)$.*

The final step for the full boundary regularity of minimizing harmonic maps is to rule out any possible boundary tangent map. For any $a \in \partial\Omega$ and $r_i \downarrow 0$, define

$$u_{a,r_i}(x) = u(a + r_i x),\ x \in r_i^{-1}(\Omega \setminus \{a\}) \to N.$$

Then (2.42) implies that after taking subsequences, $u_{a,r_i} \to \Phi$ weakly in $H^1_{\mathrm{loc}}(\mathbb{R}^n_+, N)$. $\Phi(x) = \Phi(\frac{x}{|x|})$ and $\Phi|_{\partial\mathbb{R}^n_+} = \mathrm{const}$. Moreover, a slight modification of Lemma 2.2.13 implies that u_{a,r_i} converges to Φ strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^n_+, N)$. Hence if $a \in \partial\Omega$ is a singular point of u, i.e.

$$\Theta^{n-2}_+(u, a) = \lim_{r \downarrow 0} r^{2-n} \int_{B_r(a) \cap \Omega} |\nabla u|^2 \ge \epsilon_0^2$$

then any tangent map Φ of u at a is a minimizing harmonic map on $\mathbb{R}^n_+$, which is homogeneous of degree zero and constant on $\partial\mathbb{R}^n_+$. But this contradicts Lemma 2.4.3. Therefore Theorem 2.4.1 is proven. □

Now we present an application (see [172]) of the regularity theorems we have developed for minimizing harmonic maps.

Proposition 2.4.6 *Suppose that N is compact without boundary. Any smooth map $v : S^2 \to N$ which does not extend continuously to B^3 is homotopic to a sum of smooth harmonic maps $u_j : S^2 \to N$.*

Proof. Since v is smooth, it is easy to see that $\bar{v}(x) = v\left(\frac{x}{|x|}\right) : B^3 \to N$ is an extension of v of finite energy. Let $u \in H^1(B^3, N)$ be a minimizing harmonic map with $u|_{\partial B^3} = v$. Then by Theorems 2.2.4 and 2.4.1 we have that there are finite points $x_1, \cdots, x_p$ $(p \geq 1)$ in B^3 such that $u \in C^\infty\left(\overline{B} \setminus \{x_1, \cdots, x_p\}, N\right)$. Moreover each $x_j, 1 \leq j \leq p$ is associated with a minimizing tangent map $\phi_j\left(\frac{x}{|x|}\right) : \mathbb{R}^3 \to N$, which is smooth when restricted to S^2. This proves the required result. □

To conclude this section, we add some remarks on minimizing harmonic maps among various classes of maps between manifolds.

Remark 2.4.7 (a) White [210, 211] studied the minimization problem of Dirichlet energy among homotopy classes between two Riemannian manifolds. Duzaar-Kuwert [44] studied the minimization of conformally invariant energies in homotopy class. Riviere [162] has studied minimizing 3-harmonic maps from S^3 to S^2 in the class of fixed Hopf invariant. Lin [124] studied the minimization of L^p-energy in the class of continuous maps.
(b) For $1 < p \leq n$, there have been extensive studies by many people on the partial regularity theory of minimizing p-harmonic maps. We refer interested readers to the articles by Hardt-Lin [83], and Fuchs [61].

2.5 Uniqueness of minimizing tangent maps

For a singular point $x_0 \in M$ of a minimizing harmonic map $u \in H^1(M, N)$, it follows from the previous section that for any $r_i \to 0$, there exists an energy minimizing map $\phi \in H^1_{\text{loc}}(\mathbb{R}^n, N)$ such that after taking a possible subsequence, $u(x_0 + r_i \cdot)$ converges to ϕ in $H^1_{\text{loc}}(\mathbb{R}^n, N)$. Any such a ϕ is called a *minimizing tangent map* of u at x_0. Note that by the monotonicity formula (2.20), $\phi(x) = \phi_0\left(\frac{x}{|x|}\right)$ is homogeneous of degree zero and $\phi_0 : S^{n-1} \to N$ is a weakly harmonic map. A very important question is whether a minimizing tangent map is *unique*, that is, depending on the original map u and the point x_0, but independent of the rescaling sequence $r_i \to 0$. If x_0 is an isolated singularity for u, then any minimizing tangent map ϕ of u at x_0 has 0 as its only singularity. Such uniqueness would imply that the difference $u(x) - \phi(x - x_0)$ is continuous and vanishes at x_0.
In this section, we will present the following fundamental result by Simon [181] in 1983.

Theorem 2.5.1 *Suppose $u : M \to N$ is a minimizing harmonic map, N is real analytic, and suppose that there is a minimizing tangent map ϕ of u at $x_0 \in sing(u)$ with $sing(\phi) = \{0\}$. Then ϕ is the unique minimizing tangent map of u at x_0.*

Proof. The original proof in [181] of this theorem was technically rich and interesting. Here we outline a simpler, elegant proof by Simon [182]. A crucial ingredient in the proof is an "Lojasiewcz type" inequality for the energy functional on the sphere S^{n-1}, whose proof can be found in [181].

Lemma 2.5.2 *Suppose N is real analytic and $\phi_0 \in C^\infty(S^{n-1}, N)$ is a harmonic map. Then there are $\alpha \in (0,1)$ and $\delta > 0$ depending on ϕ_0, n, and N such that*

$$|E_{S^{n-1}}(\psi) - E_{S^{n-1}}(\phi_0)|^{1-\frac{\alpha}{2}} \leq \|\tau_{S^{n-1}}(\psi)\|_{L^2(S^{n-1})} \tag{2.45}$$

for any $\|\psi - \phi_0\|_{C^3(S^{n-1})} < \delta$, where

$$\tau_{S^{n-1}}(\psi) = \Delta_{S^{n-1}}\psi + A(\psi)\left(\nabla_{S^{n-1}}\psi, \nabla_{S^{n-1}}\psi\right)$$

is the tension field of ψ from S^{n-1} to (N, h).

We now outline a proof of Theorem 2.5.1. For simplicity, assume $M = \Omega \subset \mathbb{R}^n$ and $x_0 = 0 \in \Omega$. First recall, by the definition of minimizing tangent map, that there is a seqeuence $\rho_i \downarrow 0$ such that the rescaled maps u_{ρ_i} $(\equiv u(\rho_i x))$ converge in H^1 to $\phi(x) = \phi_0\left(\frac{x}{|x|}\right)$. By the assumption, we have $\phi_0 \in C^\infty(S^{n-1}, N)$ is a smooth harmonic map. Thus for any $\eta > 0$ there is a sufficiently large j such that for $\rho = \rho_j$ we have

$$\int_{B_{\frac{3}{2}} \setminus B_{\frac{3}{4}}} |u_\rho - \phi|^2 < \eta^2. \tag{2.46}$$

Denote $\tilde{u} = u_\rho$ and keep this ρ fixed for the moment and small enough so that $B_{2\rho} \subset \Omega$, so that $\tilde{u}$ is defined on B_2. Now since ϕ is smooth in $B_{\frac{3}{2}} \setminus B_{\frac{3}{4}}$, it is clear that if $B_\sigma(z) \subset B_{\frac{3}{2}} \setminus B_{\frac{3}{4}}$ then

$$\begin{aligned} \sigma^{-n} \int_{B_\sigma(z)} |\tilde{u} - \phi(z)|^2 &\leq 2\sigma^{-n} \int_{B_\sigma(z)} \left(|\tilde{u} - \phi|^2 + |\phi - \phi(z)|^2\right) \\ &\leq 2\sigma^{-n}\eta^2 + \beta\sigma^2, \end{aligned} \tag{2.47}$$

where $\beta > 0$ is a fixed constant depending on ϕ but not depending on σ or ρ. Thus if $\gamma > 0$ is given, then for small enough η, σ (depending only on n, N, ϕ, γ) we can apply the ϵ-regularity theorem to u (see Theorem 2.2.4 and Lemma 2.2.13) on $B_\sigma(z)$ to deduce that $\|u - \phi\|_{C^3(B_\sigma(z))} \leq \gamma$. Thus we obtain for any given $\gamma > 0$ that there exists $\eta = \eta(\gamma, \phi) > 0$ such that

$$\|\tilde{u} - \phi\|_{L^2(B_{\frac{3}{2}} \setminus B_{\frac{3}{4}})} < \eta \Rightarrow \|\tilde{u} - \phi\|_{C^3(B_{\frac{5}{4}} \setminus B_{\frac{7}{8}})} < \gamma. \tag{2.48}$$

Therefore, by Lemma 2.5.2 we have

$$\left|\int_{S^{n-1}} \left(|\nabla_{S^{n-1}}\tilde{u}|^2 - |\nabla_{S^{n-1}}\phi|^2\right)\right| \leq C \|\tau_{S^{n-1}}(\tilde{u})\|_{L^2(S^{n-1})}^{\frac{2}{2-\alpha}}. \tag{2.49}$$

Now by the monotonicity identity (2.20) we have

$$2\int_{B_1} r^{2-n}\left|\frac{\partial \tilde{u}}{\partial r}\right|^2 = \int_{B_1} |\nabla \tilde{u}|^2 - \Theta(\tilde{u},0), \tag{2.50}$$

where $\Theta(\tilde{u},0) = \lim_{r\downarrow 0} r^{2-n}\int_{B_r}|\nabla\tilde{u}|^2$ satisfies

$$\Theta(\tilde{u},0) = \Theta(u,x_0) = \Theta(\phi,0) = \frac{1}{n-2}\int_{S^{n-1}} |\nabla_{S^{n-1}}\phi|^2. \tag{2.51}$$

Also, in proving (2.20) we actually showed that $\tilde{u}$ satisfies

$$(n-2)\int_{B_1}|\nabla\tilde{u}|^2 = \int_{S^{n-1}}\left(|\nabla\tilde{u}|^2 - 2|\frac{\partial\tilde{u}}{\partial r}|^2\right) \le \int_{S^{n-1}}|\nabla_{S^{n-1}}\tilde{u}|^2. \tag{2.52}$$

Hence we obtain

$$2(n-2)\int_{B_1} r^{2-n}\left|\frac{\partial\tilde{u}}{\partial r}\right|^2 \le \int_{S^{n-1}}\left(|\nabla_{S^{n-1}}\tilde{u}|^2 - |\nabla_{S^{n-1}}\phi|^2\right). \tag{2.53}$$

Writing out the harmonic map equation (1.8) for $\tilde{u}$ in terms of spherical coordinates $r=|x|$, $\omega = \frac{x}{|x|}$ gives

$$r^{1-n}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial\tilde{u}}{\partial r}\right) + r^{-2}\tau_{S^{n-1}}(\tilde{u}) + A(\tilde{u})\left(\frac{\partial\tilde{u}}{\partial r},\frac{\partial\tilde{u}}{\partial r}\right) = 0.$$

This, combined with (2.49) and (2.53), implies

$$\left(\int_{B_1} r^{2-n}\left|\frac{\partial\tilde{u}}{\partial r}\right|^2\right)^{2-\alpha} \le C\int_{S^{n-1}}\left(\left|\frac{\partial}{\partial r}(r^{n-1}\frac{\partial\tilde{u}}{\partial r})\right|^2 + \left|\frac{\partial\tilde{u}}{\partial r}\right|^2\right), \tag{2.54}$$

provided that $\|\tilde{u}-\phi\|_{L^2(B_{\frac{3}{2}}\setminus B_{\frac{3}{4}})} < \eta$.
To handle the second order derivative term, we observe that the domain deformation $\tilde{u}\left((1+t)x\right)$ is again a harmonic map. Thus $\frac{d}{dt}|_{t=0}\tilde{u}\left((1+t)x\right) = r\frac{\partial\tilde{u}}{\partial r}$ satisfies the associated Jacobi field equation:

$$\mathcal{L}\left(r\frac{\partial\tilde{u}}{\partial r}\right) = 0,$$

where $\mathcal{L}$ is the linear elliptic operator

$$\mathcal{L}v = \Delta v + 2A(\tilde{u})\left(\nabla v,\nabla\tilde{u}\right) + \frac{\partial A(u)}{\partial u}|_{u=\tilde{u}}(v)\left(\nabla\tilde{u},\nabla\tilde{u}\right) = 0.$$

Now since $\|\tilde{u}-\phi\|_{C^3(B_{\frac{5}{4}}\setminus B_{\frac{7}{8}})} \le \gamma(<< 1)$, we see that $\mathcal{L}$ has the form

$$\mathcal{L}(v) = \Delta v + b\cdot\nabla v + cv,$$

where $|b|+|c| \le \beta$ in the domain $\Omega = B_{\frac{5}{4}}\setminus B_{\frac{7}{8}}$. By the gradient estimate for $\mathcal{L}(v)=0$ (see [72]), we have

$$\sup_{B_{\frac{1}{16}}(z)} |\nabla v| \le C\,\|v\|_{L^2(B_{\frac{1}{8}}(z))}$$

for any ball $B_{\frac{1}{8}}(z) \subset B_{\frac{5}{4}} \setminus B_{\frac{7}{8}}$. Thus by covering S^{n-1} by a family of such balls $B_{\frac{1}{32}}$ we conclude

$$\sup_{S^{n-1}} \left|\nabla(r\frac{\partial \tilde{u}}{\partial r})\right|^2 \leq C \int_{B_{\frac{5}{4}} \setminus B_{\frac{7}{8}}} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2,$$

provided that $\|\tilde{u}-\phi\|_{L^2(B_{\frac{3}{2}} \setminus B_{\frac{3}{4}})} < \eta$. Therefore we obtain

$$\left(\int_{B_1} r^{2-n} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2\right)^{2-\alpha} \leq C \int_{B_{\frac{3}{2}} \setminus B_{\frac{3}{4}}} r^{2-n} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2,$$

provided that $\|\tilde{u}-\phi\|_{L^2(B_{\frac{3}{2}} \setminus B_{\frac{3}{4}})} < \eta$. By rescaling, we in fact deduce

$$\left(\int_{B_{\frac{\sigma}{2}}} r^{2-n} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2\right)^{2-\alpha} \leq C \int_{B_\sigma \setminus B_{\frac{\sigma}{2}}} r^{2-n} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2, \tag{2.55}$$

for any $\sigma \in (0,1]$ such that $\sigma^{-n}\|\tilde{u}-\phi\|^2_{L^2(B_\sigma \setminus B_{\frac{\sigma}{2}})} < \eta^2$, where C depends only on ϕ. Let $I(\sigma) = \int_{B_\sigma} r^{2-n} \left|\frac{\partial \tilde{u}}{\partial r}\right|^2$. Note that by (2.48) and (2.53) we have all $I(\sigma) \leq C\gamma$ so long as $\sigma^{-n} \|\tilde{u}-\phi\|^2_{L^2(B_\sigma \setminus B_{\frac{\sigma}{2}})} < \eta^2$, so that by making a smaller choice of γ from the start if necessary, we can assume that $I(\sigma) \leq 1$ for each σ such that $\sigma^{-n} \|\tilde{u}-\phi\|^2_{L^2(B_\sigma \setminus B_{\frac{\sigma}{2}})} < \eta^2$. Next we note that

$$0 < a < b \leq 1, \alpha \in (0,1), \beta > 0 \text{ and } a^{2-\alpha} \leq \beta\,(b-a) \Rightarrow a^{\alpha-1} - b^{\alpha-1} \geq C, \tag{2.56}$$

where $C > 0$ is a fixed constant determined only by α, β. This is easy to check by calculus, considering separately the cases when $b \geq 2a$ and $b < 2a$. Note that by applying (2.56) with $b = I(\sigma)$ and $a = I(\frac{\sigma}{2})$ we have

$$I\left(\frac{\sigma}{2}\right)^{\alpha-1} - I\left(\sigma\right)^{\alpha-1} \geq C, \tag{2.57}$$

provided that $\sigma^{-n} \|\tilde{u}-\phi\|^2_{L^2(B_\sigma \setminus B_{\frac{\sigma}{2}})} < \eta^2$.

Now note that if we use $\tilde{u}(\sigma)$ to denote the function on S^{n-1} given by $\tilde{u}(\sigma)(\omega) = \tilde{u}(\sigma\omega)$, then we have $\int_{B_{\rho_2} \setminus B_{\rho_1}} |\tilde{u}-\phi|^2 = \int_{\rho_1}^{\rho_2} \sigma^{n-1} \|\tilde{u}(\sigma)-\phi\|^2_{L^2(S^{n-1})}\, d\sigma$, hence (2.57) holds if we require

$$\|\tilde{u}(\rho)-\phi\|_{L^2(S^{n-1})} < \eta,\ \forall \rho \in \left[\frac{\sigma}{2}, \sigma\right]. \tag{2.58}$$

Now let us suppose $\sigma \in (0, \frac{1}{2}]$ is given, take the unique integer $k \geq 1$ such that $\sigma \in [2^{-k-1}, 2^{-k})$, and assume

$$\|u(s)-\phi\|_{L^2(S^{n-1})} < \eta,\ \forall s \in [\sigma, 1]. \tag{2.59}$$

Then we can apply (2.57) with $\sigma = 2^{-l}$ for $l = 0, \cdots, k$ to obtain, by summing over $l = 0, \cdots, j$,

$$I\left(2^{-j}\right)^{\alpha-1} - I(1)^{\alpha-1} \geq Cj,\ j = 1, \cdots, k.$$

For any $\tau \in \left[2^{-k}, 2^{-1}\right]$ there is an integer $2 \le j \le k$ such that $\tau \in \left[2^{-j}, 2^{-j+1}\right]$. Hence this gives

$$I(\tau) \equiv \int_0^\tau r \left\| \frac{\partial \tilde{u}}{\partial r} \right\|_{L^2(S^{n-1})} dr \le \frac{C}{|\log \tau|^{1+\beta}}, \quad \text{where } \beta = (1-\alpha)^{-1} - 1 > 0, \tag{2.60}$$

provided that (2.59) holds. Since $I(\tau)$ is a nonincreasing function of τ this then also holds for $\tau \in \left[\sigma, 2^{-k}\right]$ with a different but fixed constant C and hence (2.60) holds for all $\tau \in (\sigma, 2^{-1}]$, provided again that (2.59) holds. Note that integration by parts gives the general formula

$$\begin{aligned} \int_\sigma^1 |\log r|^{1+\frac{\beta}{2}}\, r f(r)\, dr &= |\log r|^{1+\frac{\beta}{2}} \int_0^r s f(s)\, ds \Big|_\sigma^1 \\ &\quad + \left(1 + \frac{\beta}{2}\right) \int_\sigma^1 r^{-1} |\log r|^{\frac{\beta}{2}} \int_0^r s f(s)\, ds\, dr \end{aligned}$$

and using this with $f(s) = \left\| \frac{\partial \tilde{u}}{\partial r}(s) \right\|_{L^2(S^{n-1})}$, we obtain by virtue of (2.60) that

$$\begin{aligned} \int_\sigma^1 |\log r|^{1+\frac{\beta}{2}}\, r \left\| \frac{\partial \tilde{u}}{\partial r} \right\|^2_{L^2(S^{n-1})} dr &\le C \int_\sigma^1 r^{-1} |\log r|^{\frac{\beta}{2}} I(r)\, dr \\ &\le C \int_\sigma^1 \frac{1}{r\, |\log r|^{1+\frac{\beta}{2}}}\, dr \le C, \end{aligned} \tag{2.61}$$

provided that (2.59) holds. But then we have by the Cauchy-Schwarz inequality that

$$\begin{aligned} \|\tilde{u}(\sigma) - \tilde{u}(\tau)\|_{L^2(S^{n-1})} &\le \int_\sigma^\tau \left\| \frac{\partial \tilde{u}}{\partial r} \right\|_{L^2(S^{n-1})} \\ &\le \left(\int_\sigma^\tau r |\log r|^{1+\frac{\beta}{2}} \left\| \frac{\partial \tilde{u}}{\partial r} \right\|^2_{L^2(S^{n-1})} \right)^{\frac{1}{2}} \\ &\quad \cdot \left(\int_\sigma^\tau r^{-1} |\log r|^{-1-\frac{\beta}{2}} \right)^{\frac{1}{2}} \\ &\le C |\log \tau|^{-\frac{\beta}{2}} \end{aligned} \tag{2.62}$$

for any $0 < \sigma \le \tau \le \frac{1}{2}$, provided that (2.59) holds.
Next note that by direct application of the Cauchy-Schwarz inequality

$$\begin{aligned} \|\tilde{u}(\tau) - \tilde{u}(1)\|_{L^2(S^{n-1})} &\le \int_\tau^1 \left\| \frac{\partial \tilde{u}}{\partial r} \right\|_{L^2(S^{n-1})} dr \\ &\le \left(\int_\tau^1 r^{-1}\, dr \right)^{\frac{1}{2}} \left(\int_{B_1} r^{2-n} \left| \frac{\partial \tilde{u}}{\partial r} \right|^2 \right)^{\frac{1}{2}} \equiv |\log \tau|^{\frac{1}{2}} \epsilon, \end{aligned}$$

where $\epsilon = \left(\int_{B_1} r^{2-n} |\frac{\partial \tilde{u}}{\partial r}|^2 \right)^{\frac{1}{2}}$. This, combined with the triangle inequality, implies that

$$\|\tilde{u}(\tau) - \phi\|_{L^2(S^{n-1})} < \frac{\eta}{2}, \ \forall \sigma \le \tau \le 1, \tag{2.63}$$

if

$$\epsilon|\log\sigma|^{\frac{1}{2}} < \frac{\eta}{4} \text{ and } \|\tilde{u}(1)-\phi\|_{L^2(S^{n-1})} < \frac{\eta}{4}. \tag{2.64}$$

So now suppose (2.64) holds and choose $\tau \in (\sigma, \frac{1}{2})$ such that $C|\log\tau|^{-\frac{\beta}{2}} < \frac{\eta}{4}$. Then

$$\|\tilde{u}(\sigma)-\phi\|_{L^2(S^{n-1})} \le \|\tilde{u}(\sigma)-\tilde{u}(\tau)\|_{L^2(S^{n-1})} + \|\tilde{u}(\tau)-\phi\|_{L^2(S^{n-1})}$$

and hence by (2.62) and (2.63) we deduce that $\|\tilde{u}(\sigma)-\phi\|_{L^2(S^{n-1})} < \frac{\eta}{2}$ so long as $\|\tilde{u}(s)-\phi\|_{L^2(S^{n-1})} < \eta$ for $s \in (\sigma, 1]$. Clearly this show that $\|\tilde{u}(\sigma)-\phi\|_{L^2(S^{n-1})} < \frac{\eta}{2}$ for all $\sigma \in (0,1]$ provided only we can ensure that ϵ can be selected so that (2.64) holds. However, $\tilde{u} = u_{\rho_j}$, so by choosing $\rho = \rho_j$ with j sufficiently large we can ensure both inequalities in (2.64). Hence we can apply (2.62) with any σ, τ. Then letting $\sigma = \sigma_j$ such that $\tilde{u}(\sigma_j) \to \phi$, which we can do because ϕ is a minimizing tangent map of $\tilde{u}$ at 0, we then have

$$\|\tilde{u}(\tau)-\phi\|_{L^2(S^{n-1})} \le C\,|\log\tau|^{-\frac{\beta}{2}}, \ \forall \tau \in (0, \frac{1}{2}],$$

which is the required asymptotic decay. □

Remark 2.5.3 (i) The real analyticity of N plays a crucial role for the uniqueness of minimizing tangent maps of a minimizing harmonic map at its isolated singular points. White [209] constructed a Riemannian manifold N which is not real analytic such that there exists an energy minimizing map into N that has infinitely many tangent maps at its isolated singularity.
(ii) Gulliver-White [64] constructed a real analytic Riemannian manifold N and an energy minimizing map u from M to N with an isolated singular point $x_0 \in M$ with the property that the convergence of u to its tangent map at x_0 has at most logarithmic decay.

2.6 Integrability of Jacobi fields and its applications

In this section, we will show that the logarithmic decay in the previous section can be improved to a positive power decay provided that Jacobi fields along minimizing tangent maps are integrable. Applications of the integrability of Jacobi fields were first made by Allard-Almgren [3] in the study of uniqueness of tangent cones to area minimizing surfaces, and later were explored by Simon [183, 185, 186] in the study of minimal submanifolds and minimizing harmonic maps. Here we follow §6 of the article [183] closely, except that we only discuss minimizing harmonic maps and make estimates on the punctured ball $B \setminus \{0\}$ rather than the cylinder $S^{n-1} \times [0, +\infty)$, similar to the treatment by Hardt-Lin-Wang [82] §10.3.

For simplicity, we assume $M = B \setminus \{0\}$, the punctured ball in $\mathbb{R}^n$, or $S^{n-1} \subset \mathbb{R}^n$ through this section. For a harmonic map $\phi \in C^2(M,N)$, recall that $\psi \in C^2(M, \phi^*TN)$ is called a Jacobi field along ϕ if it is a solution of the Jacobi field equation:

$$L_\phi(\psi) := \frac{d}{ds}|_{s=0}\tau(\phi_s) = 0 \text{ in } M \tag{2.65}$$

where $\phi_s \in C^1\left((-1,1), C^2(M,N)\right)$ is a C^1-family of deformation maps of ϕ such that

$$\phi_0 = \phi \text{ and } \frac{d}{ds}|_{s=0}\phi_s = \psi.$$

Direct calculations imply that (2.65) can be written as

$$L_\phi(\psi) \equiv \Delta\psi + 2A(\phi)\left(\nabla\phi, \nabla\psi\right) + \nabla_\phi A(\phi)(\psi)\left(\nabla\phi, \nabla\phi\right) = 0 \text{ in } M. \tag{2.66}$$

It is well-known that if $\phi_s \in C^1\left((-1,1), C^2(M,N)\right)$ is a family of harmonic maps, then $\psi = \frac{d}{ds}|_{s=0}\phi_s \in C^2(M, \phi^*TN)$ is a Jacobi field along ϕ. The integrablity of Jacobi fields says that the converse also holds. More precisely,

Definition 2.6.1 For a harmonic map $\phi \in C^2(M,N)$, we say that the Jacobi field equation (2.66) is *integrable*, if for any $\psi \in C^2(M, \phi^*TN)$ of $L_\phi\psi = 0$ there exists a C^1-family of smooth harmonic maps $\phi_s \in C^1\left((-1,1), C^2(M,N)\right)$ such that

$$\lim_{s\to 0}\left\|\frac{\phi_s - \phi}{s} - \psi\right\|_{C^2(M)} = 0. \tag{2.67}$$

We need the following remark (see [183] §6.1).

Remark 2.6.2 Suppose that $N \subset \mathbb{R}^L$ is real analytic and $\phi \in C^2(S^{n-1}, N)$ is harmonic such that the Jacobi field equation $L_\phi\psi = 0$ is integrable. Then for sufficiently small $\delta > 0$, there is a real analytic embedding

$$\Psi : \text{Ker}L_\phi \cap \left\{h \in C^2\left(S^{n-1}, \phi^*TN\right) : \|h\|_{C^2(S^{n-1})} < \delta\right\} \to L^2(S^{n-1})$$

with its image $\mathcal{M}$ a k-dimensional real analytic manifold ($k = \dim \text{Ker}L_\phi$) containing all solutions

$$S_\delta = \left\{\hat\phi \in C^2\left(S^{n-1}, N\right) : \tau(\hat\phi) = 0, \left\|\hat\phi - \phi\right\|_{C^2(S^{n-1})} < \frac{\delta}{2}\right\}.$$

In particular, there is $\delta_0 > 0$ such that for any $\tilde\phi \in S_{\delta_0}$

$$L_{\tilde\phi}\psi = 0 \text{ with } \|\psi\|_{C^2(S^{n-1})} < \delta_0 \quad \Rightarrow \quad \exists\hat\phi \in S_{2\delta_0} \text{ with } \hat\phi - \tilde\phi = \psi + \xi$$
$$\text{with } \|\xi\|_{C^2(S^{n-1})} \le c\,\|\psi\|^2_{C^2(S^{n-1})}, \tag{2.68}$$

where $c > 0$ depends only on ϕ, n, N. Moreover, we have

$$E_{S^{n-1}}(\tilde\phi)\left(\equiv \int_{S^{n-1}}\left|\nabla\tilde\phi\right|^2 dH^{n-1}\right) = E_{S^{n-1}}(\phi) \;\; \forall\tilde\phi \in S_{\delta_0}. \tag{2.69}$$

The main theorem of this section is the following theorem (see [183] §6, Theorem 6.6).

Theorem 2.6.3 *Assume that $N \subset \mathbb{R}^L$ is real analytic and $u \in C^2(B^n \setminus \{0\}, N)$ is a minimizing harmonic map with an isolated singular point at 0. Let $\phi(x) = \phi\left(\frac{x}{|x|}\right) \in C^\infty\left(\mathbb{R}^n \setminus \{0\}, N\right)$ be a tangent map of u at 0 such that $L_\phi\psi = 0$ is integrable. Then there exist $r_0 > 0$, $C_0 > 0$ and $\alpha \in (0,1)$ such that*

$$\left|u(x) - \phi\left(\frac{x}{|x|}\right)\right| \le C_0\,|x|^\alpha, \; \forall\,|x| \le r_0. \tag{2.70}$$

We start to indicate a proof of this theorem. First we define, for $0 < \rho < \sigma \leq 1$, the weighted C^2-norm

$$\|u\|_{\sigma,\rho} := \sup_{\rho \leq r \leq \sigma} \left(\|u\|_{L^\infty(\partial B_r)} + r\|\nabla u\|_{L^\infty(\partial B_r)} + r^2 \|\nabla^2 u\|_{L^\infty(\partial B_r)} \right).$$

For any given harmonic map $\phi \in C^2(S^{n-1}, N)$, $\epsilon > 0$ and $0 < \lambda < \frac{1}{2}$, we consider the class

$$\mathcal{Q}(\phi, \epsilon, \lambda)$$

of all minimizing harmonic maps $u : B \to N$ such that $\mathrm{sing}(u) = \{0\}$ and

$$\Theta^{n-2}(u, 0) \left(\equiv \lim_{r \downarrow 0} r^{2-n} \int_{B_r} |\nabla u|^2 \right) = \frac{E_{S^{n-1}}(\phi)}{n-2}, \ \left\| u(x) - \phi\left(\frac{x}{|x|}\right) \right\|_{1,\lambda^3} \leq \epsilon. \quad (2.71)$$

For $\mathcal{Q}(\phi, \epsilon, \lambda)$, we have the extension property that corresponding to (iii) of [183] §6.2.

Lemma 2.6.4 *For any harmonic map $\phi \in C^2(S^{n-1}, N)$, $\lambda > 0$ and $\mu > 0$ there is a positive $\hat{\epsilon} \in (0, 1)$ depending only on μ and ϕ such that*

$$\mathcal{Q}(\phi, \hat{\epsilon}, \lambda) \subset \mathcal{Q}(\phi, 1, \mu\lambda). \quad (2.72)$$

Proof. The argument was implicitly contained in the previous section. For the convenience of readers, we sketch it.

First note that if $u \in \mathcal{Q}(\phi, \hat{\epsilon}, \lambda)$ then $v(x) = u(8\lambda^3 x) \in \mathcal{Q}\left(\phi, \hat{\epsilon}, \frac{1}{2}\right)$. Hence if we verify that $v \in \mathcal{Q}\left(\phi, 1, \frac{\mu}{2}\right)$, then $u \in \mathcal{Q}(\phi, 1, \mu\lambda)$. Thus we may assume that $\lambda = \frac{1}{2}$. Suppose the conclusion were false. Then there exist a harmonic map $\phi \in C^2(S^{n-1}, N)$, $\mu \in (0, 1)$ and minimizing harmonic maps $u_i : B \to N$ such that

$$u_i \in \mathcal{Q}\left(\phi, \epsilon_i, \frac{1}{2}\right) \setminus \mathcal{Q}\left(\phi, 1, \frac{\mu}{2}\right), \text{ with } \epsilon_i \downarrow 0.$$

By the monotonicity formula (2.20) and $\Theta^{n-2}(u_i, 0) = \frac{1}{n-2} E_{S^{n-1}}(\phi)$, we have

$$\begin{aligned} 2\int_{B_1} |x|^{2-n} \left| \frac{\partial u_i}{\partial r} \right|^2 &= \int_{B_1} |\nabla u_i|^2 - \Theta^{n-2}(u_i, 0) \\ &= \int_{B_1} |\nabla u_i|^2 - \frac{1}{n-2} E_{S^{n-1}}(\phi) \\ &\leq \int_{B_1} \left| \nabla \left(u_i(\frac{x}{|x|}) \right) \right|^2 - \frac{1}{n-2} E_{S^{n-1}}(\phi) \\ &\leq \frac{1}{n-2} \left(E_{S^{n-1}}(u_i) - E_{S^{n-1}}(\phi) \right) \\ &\leq C \|u_i - \phi\|_{1,2^{-3}} \end{aligned} \quad (2.73)$$

where we have used the fact that

$$\int_{B_1} |\nabla u_i|^2 \leq \int_{B_1} \left| \nabla \left(u_i\left(\frac{x}{|x|}\right) \right) \right|^2 \text{(since } u_i \text{ is energy minimizing),}$$

and

$$E_{S^{n-1}}(u_i) - E_{S^{n-1}}(\phi) \leq C \|u_i - \phi\|^2_{C^2(S^{n-1})}, \tag{2.74}$$

since $E_{S^{n-1}}(\cdot)$ is uniformly C^2 in $\left\{v \in C^2(S^{n-1}, N) : \|v - \phi\|_{C^2(S^{n-1})} \leq \epsilon_0\right\}$ for some sufficiently small $\epsilon_0 > 0$.

If we define $u_i(\sigma)(\omega) = u_i(\sigma\omega) : S^{n-1} \to N$ then by the Cauchy-Schwarz inequality and (2.73)

$$\begin{aligned}
\|u_i(\tau) - u_i(1)\|_{L^2(S^{n-1})} &\leq \int_\tau^1 \left\|\frac{\partial u_i}{\partial r}\right\|_{L^2(S^{n-1})} dr \\
&\leq \left(\int_\tau^1 \frac{dr}{r}\right)^{\frac{1}{2}} \left(\int_{B_1} r^{2-n} \left|\frac{\partial u_i}{\partial r}\right|^2\right)^{\frac{1}{2}} \\
&\leq |\log \tau|^{\frac{1}{2}} \|u_i - \phi\|_{1,2^{-3}} \leq |\log \tau|^{\frac{1}{2}} \epsilon_i.
\end{aligned}$$

This, combined with the triangle inequality, implies that for any $\eta > 0$, if i is sufficiently large so that $\max\left\{\epsilon_i, \epsilon_i \left|\log(\frac{\mu}{4})\right|^{\frac{1}{2}}\right\} \leq \frac{\eta}{4}$ then

$$\begin{aligned}
\|u_i(\tau) - \phi\|_{L^2(S^{n-1})} &\leq \|u_i(1) - \phi\|_{L^2(S^{n-1})} + \|u_i(\tau) - u_i(1)\|_{L^2(S^{n-1})} \\
&\leq \epsilon_i \left(1 + |\log(\frac{\mu}{4})|^{\frac{1}{2}}\right) \leq \frac{\eta}{2} \text{ for all } \frac{\mu}{4} \leq \tau \leq 1.
\end{aligned} \tag{2.75}$$

Now, by the virtue of the regularity theory [171] as in the previous section, we conclude that

$$\|u_i - \phi\|_{1,\frac{\mu}{2}} \leq 1.$$

This contradicts the choices of u_i. □

Now we are ready to prove the following decay lemma (cf. also [82] Lemma 10.4 or [183] §II 6.4).

Lemma 2.6.5 *For any positive $\lambda \leq \lambda_0 = \lambda_0(n)$, there exists a positive ϵ so that if $\phi_1 \in C^2(S^{n-1}, N)$ is a harmonic map with $\|\phi_1 - \phi\|_{C^2(S^{n-1})} < \epsilon$, and if $u \in \mathcal{Q}(\phi, \epsilon, \lambda)$, then*

$$\left\|u - \phi_2\left(\frac{x}{|x|}\right)\right\|_{\lambda,\lambda^3} \leq \frac{1}{2}\left\|u - \phi_1\left(\frac{x}{|x|}\right)\right\|_{1,\lambda^2} \tag{2.76}$$

for some harmonic map $\phi_2 \in C^2(S^{n-1}, N)$.

Proof. We prove it by contradiction. Suppose that the lemma were false for some $\lambda \leq \lambda_0$. Then there exist $\epsilon_i \downarrow 0$, harmonic maps $\{\phi_i\} \subset C^2(S^{n-1}, N)$ with $\|\phi_i - \phi\|_{C^2(S^{n-1})} \leq \frac{1}{i}$, and $\{u_i\} \subset \mathcal{Q}(\phi, \epsilon_i, \lambda)$, but

$$\inf\left\{\|u_i - \tilde{\phi}\|_{\lambda,\lambda^3} : \tilde{\phi} \in C^2(S^{n-1}, N),\ \tau(\tilde{\phi}) = 0\right\} > \frac{1}{2}\|u_i - \phi_i\|_{1,\lambda^2}. \tag{2.77}$$

By Lemma 2.6.4, there is a sequence of $R_i \downarrow 0$ so that

$$\lim_{i\to 0}\left\|u_i - \phi_i\left(\frac{x}{|x|}\right)\right\|_{1,R_i} = 0. \tag{2.78}$$

Since $\|\phi_i - \phi\|_{C^2(S^{n-1})} \le \frac{1}{i}$, we have by (2.69) that for i sufficiently large,

$$E_{S^{n-1}}(\phi_i) = E_{S^{n-1}}(\phi).$$

Hence by (2.73) and (2.74), we have

$$\begin{aligned}
\int_{B_1} |x|^{2-n}\left|\frac{\partial u_i}{\partial r}\right|^2 &\le \frac{1}{2(n-2)}\left(E_{S^{n-1}}(u_i) - E_{S^{n-1}}(\phi)\right) \\
&= \frac{1}{2(n-2)}\left(E_{S^{n-1}}(u_i) - E_{S^{n-1}}(\phi_i)\right) \\
&\le C\,\|u_i - \phi_i\|^2_{1,\lambda^2}\,. \qquad (2.79)
\end{aligned}$$

This, combined with Lemma 2.6.4 and (2.78), implies that for any $R \in (0,1)$

$$\|u_i - \phi_i\|_{1,R} \le C(R)\,\|u_i - \phi_i\|_{1,\lambda^2}\,. \qquad (2.80)$$

Now we set $\beta_i = \|u_i - \phi_i\|_{1,\lambda^2}$, $w_i = \beta_i^{-1}(u_i - \phi_i)$. Then, by (2.79) and (2.80), we have

$$\|w_i\|_{1,R} \le C(R) \text{ for all } 0 < R < 1,\quad \int_{B_1} |x|^{2-n}\left|\frac{\partial w_i}{\partial r}\right|^2 \le C < +\infty \qquad (2.81)$$

for all i. Therefore we may assume that $w_i \to w$ in $C^2_{\text{loc}}(B_1 \setminus \{0\}, N)$ with

$$\|w\|_{1,R} \le C(R) \text{ for all } 0 < R < 1,\quad \int_{B_1} |x|^{2-n}\left|\frac{\partial w}{\partial r}\right|^2 < +\infty. \qquad (2.82)$$

Since by (2.78) $\beta_i \to 0$, we can check that $w \in C^2(B_1 \setminus \{0\}, \phi^*TN)$ is a Jacobi field along $\phi(\frac{x}{|x|})$ in $B_1 \setminus \{0\}$. By using the polar coordinate, it is not hard to check that $w(r,\omega) = w(r\omega)$ satisfies

$$\frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial w}{\partial r}\right) + r^{-2}L_\phi(w) = 0 \;\text{ in } B_1 \setminus \{0\}, \qquad (2.83)$$

where $L_\phi(w)$ is the Jacobi field equation 2.66 along ϕ on S^{n-1}.

Since L_ϕ is a self-adjoint, elliptic operator on S^{n-1}, there are real eigenvalues $\lambda_1 \le \lambda_2 \le \cdots$ ($\lambda_j \to \infty$ as $j \to \infty$) and corresponding eigenfunctions $\{\phi_j\} \subset C^2(S^{n-1}, \phi^*TN)$ with $(\phi_i, \phi_j)_{L^2(S^{n-1})} = \delta_{ij}$ for all i,j such that any function $v \in L^2(S^{n-1}, \phi^*TN)$ can be represented as $v = \sum_{j=1}^\infty c_j\phi_j$, with c_j real and $\sum_{j=1}^\infty c_j^2 < +\infty$.

For $0 < r < 1$, define $w(r)(\omega) = w(r\omega) : S^{n-1} \to \phi^*TN$. Then we can represent

$$w(r) = \sum_{j=1}^\infty a_j(r)\phi_j$$

and substitute it into (2.83) so that for $j \ge 1$,

$$a_j''(r) + \frac{n-1}{r}a_j'(r) - \frac{\lambda_j}{r^2}a_j(r) = 0,\; 0 < r < 1.$$

Solving this ODE, we get $a_j(r) = r^{\gamma_j}$ with

$$\gamma_j = \frac{2-n}{2} \pm \sqrt{\frac{(n-2)^2}{4} + \lambda_j}.$$

Hence we have

$$\begin{aligned} w(x) &= \sum_{j\in J_1} c_j\phi_j\left(\frac{x}{|x|}\right)|x|^{\gamma_j} + \sum_{j\in J_3}(a_j + b_j \ln|x|)\,\phi_j\left(\frac{x}{|x|}\right) \\ &+ \sum_{j\in J_2}\left(d_j\cos(\beta_j\ln|x|) + e_j\sin(\beta_j\ln|x|)\right)\phi_j\left(\frac{x}{|x|}\right)|x|^{\frac{2-n}{2}}, \quad (2.84) \end{aligned}$$

where $a_j, b_j, c_j, d_j, e_j \in \mathbb{R}$ and

$$J_1 = \left\{j : \lambda_j > -\frac{(n-2)^2}{4}\right\},\quad J_2 = \left\{j : \lambda_j < -\frac{(n-2)^2}{4}\right\} \text{ and } \beta_j = \mathrm{Im}\gamma_j,$$

and

$$J_3 = \left\{j : \lambda_j = -\frac{(n-2)^2}{4}\right\}.$$

By the second inequality in (2.82), we conclude $J_2 = \emptyset$ and $b_j = 0$ (the coefficient of $\ln|x|$) for $j \in J_3$ so that

$$w(x) = \sum_{j\in J_1} c_j\phi_j\left(\frac{x}{|x|}\right)|x|^{\gamma_j} + \sum_{j\in J_3} a_j\phi_j\left(\frac{x}{|x|}\right),\ x \in B_1\setminus\{0\}.$$

It is easy to see that $\psi(\omega) = \sum_{j\in J_3} a_j\phi_j(\omega)$ is a Jacobi field along $\phi(\omega)$ on S^{n-1}. Moreover, note that

$$\gamma_j \geq \gamma_1 = \frac{2-n}{2} + \sqrt{\frac{(n-2)^2}{4} + \lambda_1} \text{ for all } j \in J_1.$$

Hence if we define

$$S(x) \equiv \sum_{j\in J_1} c_j\phi_j\left(\frac{x}{|x|}\right)|x|^{\gamma_j} \text{ for } x \in B_1\setminus\{0\}, \quad (2.85)$$

then

$$\|S\|_{\lambda,\lambda^3} \leq \lambda^{\gamma_1}\|S\|_{1,\lambda^2} \leq \frac{1}{5}\|S\|_{1,\lambda^2} \quad (2.86)$$

provided that we choose $\lambda_0 > 0$ so that $\lambda_0^{\gamma_1} = \frac{1}{5}$.

Since $w_i \to w$ in $C^2_{\mathrm{loc}}(B_1\setminus\{0\})$ and $w(x) = S(x) + \psi$, we conclude that for i sufficiently large,

$$\left\|u_i - \phi_i\left(\frac{x}{|x|}\right) - \beta_i\left(\psi(\frac{x}{|x|}) + S\right)\right\|_{\lambda,\lambda^3} < \frac{1}{8}\beta_i. \quad (2.87)$$

Now we apply Remark 2.6.2 to conclude that there are Jacobi fields ψ_i along ϕ_i (i.e. $L_{\phi_i}(\psi_i) = 0$ on S^{n-1}) such that

$$\lim_{i\to\infty}\|\psi_i - \psi\|_{C^2(S^{n-1})} = 0,$$

and for i sufficiently large,

$$\beta_i \psi_i = \tilde{\phi}_i - \phi_i + o(\beta_i)$$

for some harmonic maps $\tilde{\phi}_i \in C^2(S^{n-1}, N)$. Therefore we obtain

$$\begin{aligned} \left\| u_i - \tilde{\phi}_i \left(\frac{x}{|x|} \right) \right\|_{\lambda, \lambda^3} &\leq \| \beta_i S \|_{\lambda, \lambda^3} + \frac{\beta_i}{4} \\ &\leq \frac{1}{5} \| \beta_i S \|_{1, \lambda^2} + \frac{\beta_i}{4} \leq \frac{1}{2} \beta_i \end{aligned}$$

for i sufficiently large. This contradicts (2.77) and completes the proof. □

It is clear that iterations of Lemma 2.6.5 yields the following theorem.

Theorem 2.6.6 *There exist positive constants $\lambda_0, \epsilon_0, \eta, C$, and μ depending only on n, ϕ so that if $0 < \lambda \leq \lambda_0$, $0 < \epsilon \leq \epsilon_0$, $\phi \in C^2(S^{n-1}, N)$ is a harmonic map, $u \in \mathcal{Q}(\phi, \epsilon, \lambda)$, and $\left\| u - \phi \left(\frac{x}{|x|} \right) \right\|_{\lambda, \lambda^2} < \eta \epsilon$, then there exists a harmonic map $\tilde{\phi} \in C^2(S^{n-1}, N)$ such that*

$$\left\| u(x) - \tilde{\phi} \left(\frac{x}{|x|} \right) \right\|_{C^2(S^{n-1})} \leq C \eta \epsilon \, |x|^{\mu} . \tag{2.88}$$

Proof. By induction on k, repeated applications of Lemma 2.6.5, we get a sequence of harmonic maps $\{\phi_k\} \subset C^2(S^{n-1}, N)$ such that

$$\left\| u(x) - \phi_{k+1} \left(\frac{x}{|x|} \right) \right\|_{\lambda^{k+1}, \lambda^{k+3}} \leq \frac{1}{2} \left\| u(x) - \phi_k \left(\frac{x}{|x|} \right) \right\|_{\lambda^k, \lambda^{k+2}},$$

and

$$\| \phi_{k+1} - \phi_k \|_{C^2(S^{n-1})} \leq C \eta \epsilon 2^{-k}.$$

Hence we deduce that (i) $\phi_k \to \tilde{\phi}$ in $C^2(S^{n-1})$ and (ii)

$$\left\| u(x) - \phi_k \left(\frac{x}{|x|} \right) \right\|_{\lambda^k, \lambda^{k+2}} \leq C \eta \epsilon 2^{-k}.$$

Thus the conclusions of this theorem follow easily. □

Proof of Theorem 2.6.3:
Note, since $\phi \in C^\infty(\mathbb{R}^n \setminus \{0\}, N)$ is a tangent map of u at 0, that for any $\epsilon > 0$ there is $\rho_0 > 0$ such that

$$\| u(\rho_0 \cdot) - \phi(\cdot) \|_{C^2(B_1 \setminus B_{\frac{1}{2}})} \leq \epsilon.$$

Hence, by considering $u_{\rho_0}(x) = u(\rho_0 x)$, we may assume that there exists $0 < \lambda = \lambda(\epsilon, n) < 1$ such that

$$\| u - \phi \|_{1, \lambda^3} \leq \epsilon. \tag{2.89}$$

Since also $\Theta^{n-2}(u,0) = \frac{1}{n-2}E_{S^{n-1}}(\phi)$, it follows $u \in \mathcal{Q}(\phi,\epsilon,\lambda)$ and hence Theorem 2.6.3 follows from theorem 2.6.6. □

It seems difficult to check the integrability criterion in higher dimensions. However, Gulliver-White [64] verified the integrability condition for harmonic maps from S^2 to any two dimensional Riemannian manifold N. Lemaire-Wood [115] verified the integrability condition for harmonic maps from S^2 to $\mathbb{CP}^2$. More precisely, the main theorem of [64] is

Theorem 2.6.7 *If dim$(N) = 2$ and $\phi \in C^2(S^2, N)$ is a harmonic map, then the Jacobi field equation $L_\phi \psi = 0$ on S^2 is integrable. In particular, if dim$(M) = 3$ and $u : M \to N$ is an energy minimizing map which is singular at $x_0 \in M$, then u converges to a unique tangent mapping u_0 at x_0 at a rate controlled by a positive power of the distance ρ from x_0.*

In their important paper [19], Brezis-Coron-Lieb have classified all minimizing tangent maps from $\mathbb{R}^3$ to S^2. More precisely, it was proved.

Theorem 2.6.8 *Suppose $\phi \in C^2(S^2, S^2)$ is a harmonic map such that $\phi\left(\frac{x}{|x|}\right) : \mathbb{R}^3 \to S^2$ is a minimizing harmonic map. Then ϕ is the restriction on S^2 of an orthogonal rotation θ of $\mathbb{R}^3$.*

Combining Theorems 2.6.8, 2.6.7 with Theorem 2.6.3, we immediately obtain the following corollary.

Corollary 2.6.9 *For any bounded domain $\Omega \subset \mathbb{R}^3$, let $u : \Omega \to S^2$ be a minimizing harmonic map which is singular at $x_0 \in \Omega$. Then there exist an orthogonal rotation θ of $\mathbb{R}^3$, $C > 0$, $\alpha \in (0,1)$ and $r_0 > 0$ such that*

$$\left| u(x_0 + x) - \theta\left(\frac{x}{|x|}\right) \right| \le C\,|x|^\alpha \quad \textit{for all } |x| \le r_0. \tag{2.90}$$

To conclude this section, we would like to mention the very important work by Simon [184] on the rectifiability of singular set of minimizing harmonic maps in dimensions at least four. In dimensions ≥ 4, higher dimensional singularities may occur for minimizing harmonic maps, and it is a great challenge to study their structure and asymptotics. In [184, 188] L. Simon has provided results on the uniqueness of the minimizing tangent maps where the tangent object may be independent of some variables and have an isolated singularity with respect to other variables (i.e., includes a whole line of singularities). Assuming a Jacobi-field integrability condition and strict minimality condition, Simon [184, 188] has been able to extend the analysis in §2.5 and §2.6 and proved the uniqueness results for such minimizing tangent maps. Furthermore, Simon [184] has proved the following remarkable theorem.

Theorem 2.6.10 *If $u : M \to N$ is minimizing harmonci map with N compact and real analytic, then for each closed ball $B \subset M$, $B \cap$ sing(u) is the union of a finite pairwise disjoint collection of locally $(n-3)$-rectfiable locally compact sets.*

There are many new ideas in the proof of the above theorem in [184], and it is beyond the scope of this book to outline them. Here we just briefly mention some key steps. First, there is a new technical criteria for $(n-3)$-rectifiability of a set involving an alternative at each scale between being located near some $(n-3)$-plane and having fixed size gaps. Second one reduces to consider singular points a at which a tangent map ϕ depends on three vairable and $\text{sing}(\phi) = \mathbb{R}^{n-3}$. The analyticity can be used to show that $\Theta^{n-2}(\phi, 0)$ only takes finitely many values $\{\alpha_1, \cdots, \alpha_l\}$. It was shown that

$$\text{sing}(u)_j = \left\{x \in \text{sing}(u) \; | \Theta^{n-2}(\phi, 0) \geq \alpha_j\right\}$$

is locally $(n-3)$-rectifiable. In the proof of this fact, Simon [184] introduced a gap measure μ which is roughly a sum of H^{n-3} restricted to the subset with no big gaps and a weighted sum of pointed masses with significant gaps. Another powerful piece of estimate is

$$\int_{T_{\theta\rho}} \psi \leq C \left(\int_{T_\rho \setminus T_{\theta\rho}} \psi \right)^{\frac{1}{2}-\alpha},$$

where

$$\psi = \int_{\text{sing}(u)_j} |x-a|^{-n} \left|(x-a) \cdot \nabla u(x)\right|^2 \, d\mu(a).$$

Here T_ρ is the union of balls with radius ρ and centered in $\text{sing}(u)_j$ in which there is an approximating $(n-3)$-plane of small tilt contained in a $\delta\rho$-neighborhood of $\text{sing}(u)_j$. The proof involves energy estimates, Lojaciewicz inequality, and L^2 estimate concerning u and its tangent map.

In a couple of special case, much can be said about the topology of the singular set. For example, Hardt-Lin [85] have proved

Theorem 2.6.11 *If $u : B^4 \to S^2$ is a minimizing harmonic map with smooth boundary data, then sing(u) consists of a finite set and finitely many Hölder continuous closed curves with only finitely many crossings.*

The proof of this theorem involves the Reifenberg topological disk lemma [159], which involes two sides approximation of $\text{sing}(u)$ by lines. The fact that $\text{sing}(u)$ is locally in a small neighborhood of some lines follows from the small energy regularity theorem. The opposite fact that the approximating line is in a small neighborhood of $\text{sing}(u)$ is by a topological obstruction in the case $M = B^4$ and $N = S^2$.

Very recently, Lin-Wang [132] have been able to extend Theorem 2.6.11 to stable-stationary harmonic maps from B^5 to S^3. More precisely, we have

Theorem 2.6.12 *If $u : B^5 \to S^3$ is a stable-stationary harmonic map with smooth boundary data. Then sing(u) consists of a finite set and finitely many Hölder continuous closed curves with only finitely many crossings.*

Chapter 3

Regularity of stationary harmonic maps

In this chapter, we will present the regularity theorem of weakly harmonic maps in dimension two, due to Hélein [90, 91, 92], and the partial regularity theorem of stationary harmonic maps in higher dimensions, due to Evans [45] and Bethuel [11]. We will also present some optimal partial regularity theorems on stable-stationary harmonic maps into spheres, due to Hong-Wang [98] and Lin-Wang [132]. The interested readers can consult with the paper by Chang-Wang-Yang [24] for an alternative proof of the main theorem in [45]. We would like to remark that Riviére [161] has found a set of new conservation law for harmonic maps in dimensions two and given a new proof of the main theorem in [92], subsequentially Riviére-Struwe [163] have found simplifications and improvement on the regularity theory of stationary harmonic maps by [11] in higher dimensions.

This chapter is organized as follows. In §3.1, we present the classical theorem on weakly harmonic maps into regular balls by Hildebrandt-Kaul-Widman [95]. In §3.2, we present the regularity of weakly harmonic maps by Hélein [90, 91, 92]. In [93], Hélein has given a detailed account of his important works. In §3.3, we present the partial regularity of stationary harmonic maps. In §3.4, we present some optimal partial regularity for stable-stationary harmonic maps.

3.1 Weakly harmonic maps into regular balls

In this section, we will present the fundamental theorem, due to Hildebrandt-Kaul-Widman [95], on the regularity of weakly harmonic maps whose images are contained in any regular ball of N. The presentation here follows the book by Jost [102] Chapter 4 closely. We begin with

Definition 3.1.1 For $q \in N$, let $B_r(q) = \{p \in N : \text{dist}_N(p, q) \le r\}$ be the geodesic ball in N with center q and radius r, and let $C(q)$ be the cut locus of q. We say that $B_r(q)$ is a *regular ball* in N if
(i)$\sqrt{\kappa} r < \frac{\sqrt{\pi}}{2}$, where $\kappa = \max\left\{0, \sup_{B_r(q)} K^N\right\}$,
(ii) $C(q) \cap B_r(q) = \emptyset$.

Note that by virtue of a theorem of Hadamard (cf. [63]), if N is simply connected and $K^N \le 0$, then each ball in N is a regular ball.

The crucial property of regular balls we need is the following proposition (see [95] or [102]).

Proposition 3.1.2 *For a regular ball $B_r(q) \subset N$, we have $dist^2_N(y,q) \in C^2(B_r(q))$ is a strictly convex function, and any pair of points $y_1, y_2 \in B_r(q)$ can be connected by a unique geodesic contained in $B_r(q)$ which contains no pair of conjugate points.*

A main result of this section is the regularity theorem (cf. [95]).

Theorem 3.1.3 *Let (M,g) be a compact Riemannian manifold without boundary and $\Omega \subset M$ be a domain. Let $B_r(p) \subset N$ be a regular ball and $u \in H^1(\Omega, B_r(p))$ be a weakly harmonic map. Then for any $x_0 \in \Omega$ and $\delta > 0$, there is $\rho > 0$ such that*

$$osc_{B_\rho(x_0)} u \le \delta,$$

where ρ depends only on δ, $dist(x_0, \partial\Omega)$, the injectivity radius of Ω, M, and the curvature bounds of N on $B_r(p)$. In particular, u is smooth in Ω.

To simplify the presentation, we assume that $(M,g) = (\mathbb{R}^n, dx^2)$ and $\Omega \subset \mathbb{R}^n$ is a bounded domain. There are two critical steps. The first step is the following lemma.

Lemma 3.1.4 *Suppose that $u : \Omega \to Y \subset N$ is a weakly harmonic map and $f \in C^2(Y)$ is a strictly convex function on $u(\Omega)$. Then for any $\epsilon > 0$ and $0 < R_0 < dist(x_0, \partial\Omega)$, there is $R_1 > 0$ depending only on f and Ω such that for some $R \in [R_1, R_0]$, we have*

$$R^{2-n} \int_{B_R(x_0)} |\nabla u|^2 \le \epsilon. \tag{3.1}$$

Proof. Set $h = f \circ u$. By the chain rule of harmonic maps (cf. [102]), there is $\lambda > 0$ depending only on f such that

$$\Delta h \ge \lambda |\nabla u|^2. \tag{3.2}$$

Assume $x_0 = 0 \in \Omega$. Let

$$g_\rho(x) = \min\left\{ |x|^{2-n} - \rho^{2-n},\ (\frac{\rho}{2})^{2-n} - \rho^{2-n} \right\} \text{ on } B_\rho.$$

Then, since $g_\rho = 0$ on ∂B_ρ and $\Delta g_\rho = 0$ in $B_\rho \setminus B_{\frac{\rho}{2}}$, we have

$$\begin{aligned} \lambda \int_{B_\rho} |\nabla u|^2 g_\rho &\le \int_{B_\rho} \Delta h g_\rho = -\int_{B_\rho \setminus B_{\frac{\rho}{2}}} \langle \nabla h, \nabla g_\rho \rangle \\ &= -\int_{\partial(B_\rho \setminus B_{\frac{\rho}{2}})} h \frac{\partial g_\rho}{\partial \nu} \\ &= \frac{n-2}{\rho^{n-1}} \int_{\partial B_\rho} h - \frac{n-2}{(\frac{\rho}{2})^{n-1}} \int_{\partial B_{\frac{\rho}{2}}} h. \end{aligned} \tag{3.3}$$

Denote $g_i = g_{\frac{R_0}{2^i}}$ for all $i \geq 0$. Then we have

$$|x|^{2-n} \leq \left(\frac{R_0}{2^{i+1}}\right)^{2-n} \quad \text{and } g_i \geq \left(1 - 2^{2-n}\right) |x|^{2-n} \text{ on } B_{\frac{R_0}{2^i}} \setminus B_{\frac{R_0}{2^{i+1}}}.$$

Thus we have

$$\begin{aligned}
\int_{B_{R_0}} |x|^{2-n}|\nabla u|^2 &\leq 2^{n-2}\sum_{i=0}^{\infty}\left(\frac{R_0}{2^i}\right)^{2-n}\int_{B_{\frac{R_0}{2^i}}\setminus B_{\frac{R_0}{2^{i+1}}}} |\nabla u|^2 \\
&\leq c_n \sum_{i=0}^{\infty}\int_{B_{\frac{R_0}{2^i}}} g_i|\nabla u|^2
\end{aligned}$$

where c_n depends only on n. Therefore, applying (3.3), we have

$$\begin{aligned}
&\int_{B_{R_0}} |x|^{2-n}|\nabla u|^2 \\
&\leq \frac{c_n}{\lambda}\sum_{i=0}^{\infty}\left\{\left(\frac{R_0}{2^i}\right)^{1-n}\int_{\partial B_{\frac{R_0}{2^i}}} h - \left(\frac{R_0}{2^{i+1}}\right)^{1-n}\int_{\partial B_{\frac{R_0}{2^{i+1}}}} h\right\} \\
&\leq \frac{c_n}{\lambda}{R_0}^{1-n}\int_{\partial B_{R_0}} h \leq c(n,\lambda)\|f\|_{L^\infty(Y)}. \qquad (3.4)
\end{aligned}$$

Denote $\mu_i = \left(\frac{R_0}{2^i}\right)^{1-n}\int_{\partial B_{\frac{R_0}{2^i}}} h$ for all $i \geq 0$. Then, since h is sub harmonic, $\mu_i \geq \mu_{i+1}$ for all $i \geq 0$. From (3.4), we have

$$\sum_{i=0}^{\infty}(\mu_i - \mu_{i+1}) \leq c(n,\lambda)\|f\|_{L^\infty(Y)} < +\infty$$

so that $\lim_{i\to\infty}(\mu_i - \mu_{i+1}) = 0$. Therefore, for any $\epsilon > 0$ there exists a sufficiently large $i_0 = i_0(\epsilon)$ depending only on f and Ω such that $\mu_{i_0} - \mu_{i_0+1} \leq \epsilon$. This, combined with (3.3), implies that (3.1) holds with $R_1 = \frac{R_0}{2^{i_0}}$. □

Next we have

Lemma 3.1.5 *Under the same assumptions as in Lemma 3.1.4, there exist $R_2 = R_2(\epsilon) > 0$ and $R \in [R_2(\epsilon), R_0]$ such that if $B_{2R_0}(x_0) \subset \Omega$, then*

$$f(u(y)) \leq f(u_{x_0,R}) + 4\epsilon - c_n\int_{B_R(x_0)} |x-y|^{2-n}\nabla^2 f(u)(\nabla u, \nabla u)(x)\,dx, \qquad (3.5)$$

holds for all $y \in B_R(x_0)$, where c_n depends only on n, and $u_{x_0,R}$ is given by

$$u_{x_0,R} = \frac{1}{|T_R(x_0)|}\int_{T_R(x_0)} u, \quad \textit{where } T_R(x_0) = B_{2R}(x_0) \setminus B_R(x_0).$$

Proof. Assume $x_0 = 0$ and denote $T_R = T_R(x_0)$. Let $\eta \in \mathrm{Lip}(\Omega)$ be such that $\mathrm{supp}(\eta) \subset B_{2R}$, $0 \le \eta \le 1$, $\eta \equiv 1$ on B_R, and $|\nabla \eta| \le CR^{-1}$. Fixed $y \in B_{\frac{R}{2}}$. For small $\delta > 0$, define $g_\delta : B_{2R} \to \mathbb{R}$ by

$$g_\delta(x) = \begin{cases} |x-y|^{2-n} & \text{if } |x-y| \ge \delta, \\ \delta^{2-n} & \text{if } |x-y| \le \delta. \end{cases}$$

Since $\Delta u \perp T_u N$ in the distribution sense and $\nabla f(u) \eta g_\delta \in T_u N$, we have

$$\begin{aligned} \int_{B_{2R}} \nabla(\eta f(u)) \cdot \nabla g_\delta &= -\int_{B_{2R}} \eta g_\delta \nabla^2 f(\nabla u, \nabla u) - \int_{B_{2R}} g_\delta \nabla \eta \cdot \nabla(f(u)) \\ &\quad + \int_{B_{2R}} f(u) \nabla \eta \cdot \nabla g_\delta \\ &= I_\delta + II_\delta + III_\delta. \end{aligned} \tag{3.6}$$

It is easy to see that the left hand side of (3.6) can be estimated by

$$\begin{aligned} \int_{B_{2R}} \nabla(\eta f(u)) \cdot \nabla g_\delta &= \int_{B_{2R} \setminus B_\delta(y)} \nabla(\eta f(u)) \cdot \nabla g_\delta \\ &= \int_{\partial(B_{2R} \setminus B_\delta(y))} \eta f(u) \frac{\partial g_\delta}{\partial \nu} \\ &= \int_{\partial B_\delta(y)} \eta f(u) \frac{\partial g_\delta}{\partial |x-y|} \\ &= -(n-2)\delta^{1-n} \int_{\partial B_\delta(y)} \eta f(u) \\ &\to -(n-2)\omega_n f(u(y)), \text{ as } \delta \to 0, \end{aligned} \tag{3.7}$$

where $\omega = H^{n-1}(S^{n-1})$ is the volume of the sphere S^{n-1}.

Note that since $y \in B_{\frac{R}{2}}$,

$$\int_{T_R} \eta g_\delta \nabla^2 f(u)(\nabla u, \nabla u) \le CR^{2-n} \int_{B_{2R}} |\nabla u|^2 \le \epsilon$$

holds for some R, $R_1(\epsilon) \le R < R_0 \left(= \frac{1}{2}\mathrm{dist}(x_0, \partial\Omega)\right)$, where $R_1(\epsilon)$ is given by Lemma 3.1.4. Hence we have

$$\begin{aligned} \lim_{\delta \to 0} I_\delta &= -\lim_{\delta \to 0} \left\{ \int_{B_R} + \int_{T_R} \right\} \eta g_\delta \nabla^2 f(u)(\nabla u, \nabla u) \\ &\ge -\int_{B_R} \eta |x-y|^{2-n} \nabla^2 f(u)(\nabla u, \nabla u) - \epsilon. \end{aligned} \tag{3.8}$$

Since $\nabla \eta = 0$ outside T_R, it is easy to see

$$\begin{aligned} \lim_{\delta \to 0} |II_\delta| &\le \lim_{\delta \to 0} CR^{-1} \int_{T_R} g_\delta |\nabla u| \le CR^{1-n} \int_{T_R} |\nabla u| \\ &\le C \left(R^{2-n} \int_{B_{2R}} |\nabla u|^2 \right)^{\frac{1}{2}} \le \epsilon \end{aligned} \tag{3.9}$$

for some R chosen as in above, where we have used the fact that

$$g_\delta(x) \le \left(\frac{3R}{2}\right)^{2-n} \quad \text{on } T_R,$$

since $y \in B_{\frac{R}{2}}$.

To estimate III_δ, we write $f(u) = f(u_R) + (f(u) - f(u_R))$ and observe that

$$\int_{B_{2R}} \nabla\eta \cdot \nabla g_\delta = \int_{B_{2R}\setminus B_\delta(y)} \nabla\eta \cdot \nabla g_\delta = \int_{\partial B_\delta(y)} \frac{\partial g_\delta}{\partial |x-y|} \to -(n-2)\omega_n,$$

as $\delta \to 0$. Hence we obtain

$$\begin{aligned}
\lim_{\delta\to 0} III_\delta &\le -(n-2)\omega_n f(u_R) + \int_{T_R} (f(u) - f(u_R))\, \nabla|x-y|^{2-n} \cdot \nabla\eta \\
&\le -(n-2)\omega_n f(u_R) + CR^{-n} \int_{T_R} |f(u) - f(u_R)| \\
&\le -(n-2)\omega_n f(u_R) + C\|\nabla f\|_{L^\infty} \left(R^{-n} \int_{T_R} |u - u_R|^2\right)^{\frac{1}{2}} \\
&\le -(n-2)\omega_n f(u_R) + C\left(R^{2-n} \int_{T_R} |\nabla u|^2\right)^{\frac{1}{2}} \\
&\le -(n-2)\omega_n f(u_R) + \epsilon, \qquad (3.10)
\end{aligned}$$

where we have used both Hölder inequality and Poincaré inequality.

Putting together (3.6), (3.7), (3.8), (3.9) and (3.10), we obtain (3.5). □

Proof of Theorem 3.1.3:
Let

$$h_0 = \min\left\{\frac{\pi}{2r\sqrt{\kappa}} - 1, 1\right\}$$

and $\epsilon_1 \in (0,1)$ be such that

$$h_1 = \frac{\pi}{2\sqrt{\kappa}} - \left\{(1-h_0)^2 r^2 + \epsilon_1\right\}^{\frac{1}{2}} > 0.$$

Let

$$h = \min\left\{\frac{2h_1\sqrt{\kappa}}{\pi}, h_0\right\}, \quad \text{and } 0 < \epsilon_2 \le \delta^2.$$

Choose ϵ in Lemma 3.1.4 by

$$\epsilon = \frac{1}{8} h(2-h) \min\left\{\epsilon_1, \frac{\epsilon_2}{8}\right\}$$

and let s be the smallest positive integer such that

$$(1-h)^{2s} < \frac{\epsilon_2}{8r^2}.$$

Let's start with $R_0 = \frac{1}{2}\text{dist}(x_0, \partial\Omega)$ and $p_0 = p$. Let c_0 be the unique geodesic from p_0 to u_{R_0}, and let p_1 be the unique point in c_0 such that

$$\text{dist}(p_1, p_0) = h_0 \text{dist}(u_{x_0,R_0}, p_0).$$

Then for any $q \in B_r(p)$, we have

$$\text{dist}(q, p_1) \le \text{dist}(q, p_0) + \text{dist}(p_1, p_0) \le (1 + h_0)r < \frac{\pi}{2\sqrt{\kappa}}.$$

Hence, by Proposition 3.1.2, $\text{dist}^2(q, p_1)$ is strictly convex on $B_r(p)$. Then, by Lemma 3.1.4 and Lemma 3.1.5, we have that for any $y \in B_{R_1}(x_0)$, with $2R_1$ the radius $R \le R_0$ such that (3.5) holds, (3.5) implies

$$\begin{aligned}
\text{dist}^2\left(u(y), p_1\right) &\le \text{dist}^2\left(u_{x_0,R_0}, p_1\right) + 4\epsilon \\
&\le (1 - h_0)^2 \sup_{x \in B_{2R_0}(x_0)} \text{dist}^2\left(u(x), p_1\right) + 4\epsilon. \qquad (3.11)
\end{aligned}$$

Now we can iterate (3.11) as follows. Let $j \in \mathbb{N}$, suppose that there are $p_i \in B_r(p)$ and radius R_i for $i \le j - 1$ with the property that for $y \in B_{R_i}(x_0)$

$$\text{dist}^2\left(u(y), p_i\right) \le (1 - h_0)^2 \sup_{x \in B_{2R_0}(x_0)} \text{dist}^2\left(u(x), p_1\right) + \epsilon_1^2 \qquad (3.12)$$

and

$$\text{dist}^2\left(u(y), p_i\right) \le (1 - h)^2 \sup_{x \in B_{2R_i}(x_0)} \text{dist}^2\left(u(x), p_{i-1}\right) + 4\epsilon. \qquad (3.13)$$

Then we want to show (3.12) and (3.13) hold for $i = j$ and some $p_j \in B_r(p)$ and radius R_j.

First, by (3.12) we have

$$\text{dist}^2\left(u(y), p_{j-1}\right) \le \frac{\pi}{2\sqrt{\kappa}} - h_1 \quad \text{for } y \in B_{R_{j-1}}(x_0).$$

Note that, since $\text{dist}\left(u_{x_0,R_{j-1}}, p_{j-1}\right) < \frac{\pi}{2\sqrt{\kappa}}$, there exists a unique geodesic $c_{j-1} \subset B_r(p)$ from p_{j-1} to $u_{x_0,R_{j-1}}$. Let $p_j \in c_{j-1}$ be the point such that

$$\text{dist}\left(p_j, p_{j-1}\right) = h\text{dist}\left(u_{x_0,R_{j-1}}, p_{j-1}\right).$$

Then for $y \in B_{R_{j-1}}(x_0)$,

$$\begin{aligned}
\text{dist}\left(u(y), p_j\right) &\le \text{dist}\left(u(y), p_{j-1}\right) + \text{dist}\left(p_j, p_{j-1}\right) \\
&\le \left((1 - h_0)^2 r^2 + \epsilon_1\right)^{\frac{1}{2}} + hr \\
&\le \frac{\pi}{2\sqrt{\kappa}} - h_1 + hr < \frac{\pi}{2\sqrt{\kappa}}.
\end{aligned}$$

Hence, $\text{dist}\left(u(y), p_j\right)$ is strictly convex on $B_{R_{j-1}}(x_0)$, and from (3.13) we have that there is $R_j \le \frac{R_{j-1}}{2}$ such that for $y \in B_{R_j}(x_0)$

$$\begin{aligned}
\text{dist}^2\left(u(y), p_j\right) &\le \text{dist}^2\left(u(y), p_{j-1}\right) + 4\epsilon \\
&\le (1 - h)^2 \text{dist}^2\left(u_{x_0,R_{j-1}}, p_{j-1}\right) + 4\epsilon \\
&\le (1 - h)^2 \sup_{x \in B_{2R_j}(x_0)} \text{dist}^2\left(u(x), p_{j-1}\right) + 4\epsilon
\end{aligned}$$

Thus (3.13) holds for $i = j$. Iterating (3.13), we obtain

$$\sup_{y\in B_{R_j}(x_0)} \operatorname{dist}^2(u(y), p_j) \le (1-h)^{2j} \sup_{B_{R_0}(x_0)} \operatorname{dist}^2(u(y), p_0) + \frac{4\epsilon}{1-(1-h)^2}.$$

For $j \ge 1$, $\frac{1}{1-(1-h)^2} + (1-h)^{2j} \le \frac{2}{h(2-h)}$. Thus we have

$$\sup_{y\in B_{R_j}(x_0)} \operatorname{dist}^2(u(y), p_j) \le (1-h)^{2j} r^2 + \min\left\{\epsilon_1, \frac{\epsilon_2}{8}\right\}. \tag{3.14}$$

In particular, (3.12) holds for $i = j$. Moreover, (3.14) implies

$$\left(\operatorname{osc}_{B_{R_j}(x_0)} u\right)^2 \le 4(1-h)^{2j} r^2 + \frac{\epsilon_2}{2} \le \epsilon_2$$

and hence $\operatorname{osc}_{B_{R_j}(x_0)} u \le \delta$. Since $\delta > 0$ is arbitrarily small, this implies that u is continuous in Ω. Then by the hole filling argument similar to Chapter 2.1, we can show $u \in C^\alpha(\Omega, N)$ for some $\alpha \in (0,1)$. Then by the bootstrap argument, we can finally prove $u \in C^\infty(\Omega, N)$ (see also [102] Chapter 4.5). This completes the proof. □

Weakly harmonic maps from a Riemannian manifold with boundary, with smooth Dirichlet boundary data, into regular balls are also smooth near the boundary. In fact, we have the following theorem, due to Giaquinta-Hildebrandt [68], whose proof can also be found in [102] Theorem 4.7.1.

Theorem 3.1.6 *Suppose $u : \Omega \subset M \to B_r(p) \subset N$ is a weakly harmonic map, where $B_r(p) \subset N$ is a regular ball. Suppose $\partial\Omega$ is of class C^2, and the sectional curvature $|K^M| \le \Lambda^2$ on Ω. For any given $g(= u|_{\partial\Omega}) \in C^0(\partial\Omega, N)$, then for any $\epsilon > 0$ there is $\delta > 0$, depending on ϵ, M, Λ, Ω, N and the modulus of continuity of g, such that for any $x_0 \in \partial\Omega$,*

$$\mathit{dist}(u(y), u(x_0)) \le \epsilon \quad \text{for } y \in B_\delta(x_0) \cap \Omega. \tag{3.15}$$

If $g \in C^\alpha(\Omega, N)$ for some $\alpha \in (0,1)$, then there $\beta \in (0,1)$ and $c_\beta > 0$ depending on α, M, Λ, Ω, N and $\|g\|_{C^\alpha}$ such that for any $x_0 \in \partial\Omega$

$$\mathit{dist}(u(y), u(x_0)) \le c_\beta |y - x_0|^\beta \quad \text{for } y \in B_\delta(x_0) \cap \Omega. \tag{3.16}$$

In particular, if $\partial\Omega$ is of class C^∞ and $g \in C^\infty(\partial\Omega, N)$, then $u \in C^\infty(\overline{\Omega}, N)$.

Using both theorem 3.1.3 and 3.1.6, Hildebrandt-Kaul-Widman [95] have established the existence of smooth harmonic maps with given boundary data contained in a regular ball that admit an extension with finite energy. More precisely,

Theorem 3.1.7 *Suppose that $B_r(p) \subset N$ is a regular ball and $\Omega \subset M$ is a bounded domain and $g : \Omega \to B_r(p)$ has finite energy. Then there exists a harmonic map $u \in C^{2,\alpha}(\Omega, N)$ with $u|_{\partial\Omega} = g$. Moreover, at $\partial\Omega$, u is as regular as g and $\partial\Omega$ permit.*

To prove this theorem, we first need the following maximum principle for minimizing harmonic maps (cf. [102] Lemma 4.10.1).

Lemma 3.1.8 *Suppose that* B_0 *and* B_1*,* $B_0 \subset B_1$*, are closed subsets of* N*, and* $\pi : B_1 \to B_0$ *is a* C^1*-retraction map such that*

$$|\nabla\pi(v)| < |v| \text{ if } 0 \neq v \in T_x N, \ x \in B_1 \setminus B_0.$$

If $h : \Omega \to B_1$ *is an energy minimizing map with a given boundary value* $g : \partial\Omega \to B_0$*, then* $h(\Omega) \subset B_0$*.*

Proof. Since $\pi|_{B_0}$ is the identity map, we have $(\pi \circ h)(\Omega) \subset B_0$ and $\pi \circ h|_{\partial\Omega} = h|_{\partial\Omega}$. Hence the minimality of h implies

$$\begin{aligned}\int_\Omega |\nabla h|^2 &\leq \int_\Omega |\nabla(\pi \circ h)|^2 \\ &= \int_{\Omega_1} |\nabla h|^2 + \int_{\Omega_2} |\nabla\pi(h)|^2 |\nabla h|^2 \\ &< \int_{\Omega_1} |\nabla h|^2 + \int_{\Omega_2} |\nabla h|^2 = \int_\Omega |\nabla h|^2\end{aligned}$$

if Ω_2 has positive Lebesgue measure, where $\Omega_1 = \{x \in \Omega \mid h(x) \in B_0\}$ and $\Omega_2 = \Omega \setminus \Omega_1$. Therefore Ω_2 must have zero Lebesgue measure and $h(\Omega) \subset B_0$. □

Proof of Theorem 3.1.7:
Let $r_1 \in \left(r, \frac{\pi}{2\sqrt{\kappa}}\right)$ be such that $B_{r_1}(p) \subset N$ is also a regular ball. Consider

$$\min_{v\in V} \int_\Omega |\nabla v|^2, \text{ where } V \equiv \left\{v \in H^1(\Omega, B_{r_1}(p)), v|_{\partial\Omega} = g\right\}.$$

By the direct method, there is a minimizing harmonic map $u \in V$.

Let $B_0 = B_r(p)$ and $B_1 = B_{r_1}(p)$. Then $B_0 \subset B_1$. Moreover, there exists a C^1-retraction map $\pi : B_1 \to B_0$ that is distance decreasing in $B_1 \setminus B_0$. Hence the assumptions of Lemma 3.1.8 are satisfied. Therefore $u(\Omega) \subset B_0$. We claim that u is a weakly harmonic map. In fact, since for any $\eta \in C_0^1(\Omega, \mathbb{R}^l)$ ($l = \dim(N)$) $(u + t\eta)(\Omega) \subset B_{r_1}(p)$ for $|t|$ sufficiently small, we have

$$\frac{d}{dt}\Big|_{t=0} \int_\Omega |\nabla(u + t\eta)|^2 = 0.$$

Finally we can apply Theorems 3.1.3 and 3.1.6 to deduce both the interior and boundary regularity of u. □

Remark 3.1.9 If $B_r(p) \subset N$ is a regular ball, $g \in C^0(\overline{\Omega}, B_r(p)) \cap H^1(\Omega, B_r(p))$, then there is a unique harmonic map $u : \Omega \to B_r(p)$ with $u|_{\partial\Omega} = g$, which is also a minimizing harmonic map. This is a consequence of the uniqueness theorem by Jäger-Kaul [108].

3.2 Weakly harmonic maps in dimension two

For domain dimension $n = 1$, the regularity of a weakly harmonic map is immediate from the geodesic ODE. For $n = 2$, the problem is much more difficult. Here the regularity was established for energy minimizers by Morrey [145] in 1948, for conformal harmonic maps by Grüter [75] in 1981, for stationary harmonic maps by Schoen [166] in 1983 and finally for general weakly harmonic maps by Hélein [90, 91, 92] in 1991. In the section, we will present a proof of Hélein's regularity theorem. The interested readers should also consult with [93] where many powerful analytic techniques were presented.

Theorem 3.2.1 *For a Riemannian surface M with $\partial M = \emptyset$, if $u \in H^1(M, N)$ is a weakly harmonic map then $u \in C^\infty(M, N)$.*

Hélein [90] first discovered this theorem for a round spherical target manifold $N = S^{L-1} \subset \mathbb{R}^L$, in which he made a crucial observation that *the nonlinearity $|\nabla u|^2 u$ exhibits a Jacobian determinant structure* and hence belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ through earlier results by Coifman-Lions-Meyers-Semmes [35], a much better space than the usual $L^1(\mathbb{R}^2)$. More precisely,

Lemma 3.2.2 *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If $u \in H^1(\Omega, S^{L-1})$ is a weakly harmonic map, then for any $1 \le i, j \le L$, $V^{ij} \equiv \nabla u^i u^j - \nabla u^j u^i \in L^2(\Omega, \mathbb{R}^n)$, is divergenc free (i.e., $\mathrm{div}(V^{ij}) = 0$ in the distribution sense). Moreover, u satisfies*

$$- \Delta u^i = \sum_{j=1}^{n} \nabla u^j \cdot V^{ij}, \ 1 \le i \le L. \tag{3.17}$$

Proof. Since $\Delta u + |\nabla u|^2 u = 0$, direct calculations give

$$\begin{aligned} \mathrm{div}(V^{ij}) &= \mathrm{div}(\nabla u^i u^j - \nabla u^j u^i) = \Delta u^i u^j - \Delta u^j u^i \\ &= |\nabla u|^2 u^j u^i - |\nabla u|^2 u^i u^j = 0. \end{aligned}$$

This implies V^{ij} is divergence free. To see (3.17), observe that since $|u|^2 = 1$, we have $\sum_{j=1}^{L} u^j \nabla u^j = 0$. Therefore we have

$$\begin{aligned} -\Delta u^i &= |\nabla u|^2 u^i = \sum_{j=1}^{L} \nabla u^j \cdot (u^i \nabla u^j) \\ &= \sum_{j=1}^{L} \nabla u^j \cdot (u^i \nabla u^j - u^j \nabla u^i) = \sum_{j=1}^{L} \nabla u^j \cdot V^{ij}. \end{aligned}$$

This completes the proof. □

Note that the right hand of (3.17) has a special structure, i.e., the product of a curl free vector field and a div free vector field, and it belongs to $\mathcal{H}^1$, the Hardy space, which we now define.

Definition 3.2.3 A function $f \in L^1(\mathbb{R}^n)$ belongs to the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ if

$$f^*(x) = \sup_{t>0} |(\eta_t * f)(x)| \in L^1(\mathbb{R}^n), \tag{3.18}$$

where $\eta \in C_0^\infty(\mathbb{R}^n)$ is a function such that $\int_{\mathbb{R}^n} \eta = 1$, $\eta_t(x) = t^{-n}\eta\left(\frac{x}{t}\right)$ for $t > 0$, and

$$(\eta_t * f)(x) = \int_{\mathbb{R}^n} \eta_t(x-y) f(y)\, dy = \int_{\mathbb{R}^n} \eta_t(y) f(x-y)\, dy$$

is the convolution of f by η_t. We equip $\mathcal{H}^1(\mathbb{R}^n)$ with the norm

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} = \|f\|_{L^1(\mathbb{R}^n)} + \|f^*\|_{L^1(\mathbb{R}^n)}. \tag{3.19}$$

The following is the Hardy space analogue of Calderón-Zygmund L^p-theory [192] $(1 < p < +\infty)$.

Theorem 3.2.4 *For $f \in \mathcal{H}^1(\mathbb{R}^n)$, let $\phi \in L^1(\mathbb{R}^n)$ be a solution of*

$$-\Delta\phi = f \text{ in } \mathbb{R}^n.$$

Then $\nabla^2\phi \in L^1(\mathbb{R}^n)$ and

$$\left\|\nabla^2\phi\right\|_{L^1(\mathbb{R}^n)} \le C \|f\|_{\mathcal{H}^1(\mathbb{R}^n)}. \tag{3.20}$$

We also have the following theorem due to [35].

Theorem 3.2.5 *For $1 < p < +\infty$ and $q = \frac{p}{p-1}$, suppose $h \in W^{1,p}(\mathbb{R}^n)$ and $G \in L^q(\mathbb{R}^n, \mathbb{R}^n)$ is a divergence free vector field. Then the function $f := \nabla h \cdot G \in \mathcal{H}^1(\mathbb{R}^n)$. Moreover, there exists a positive constant C_n depending only on n such that*

$$\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C_n \|h\|_{W^{1,p}(\mathbb{R}^n)} \|G\|_{L^q(\mathbb{R}^n)}. \tag{3.21}$$

For the need of later development, we recall the definition of Lorentz spaces on $\mathbb{R}^n$ (cf. [93, 198]).

Definition 3.2.6 Let $\Omega \subset \mathbb{R}^n$ be an open subset, $p \in (1, +\infty)$, $q \in [1, +\infty]$. The Lorentz space $L^{(p,q)}(\Omega)$ is the set of measurable functions $f : \Omega \to \mathbb{R}$ such that

$$\|f\|_{L^{(p,q)}(\Omega)} = \begin{cases} \left(\int_0^{+\infty} (t^{\frac{1}{p}} f^*(t))^q \frac{dt}{t}\right)^{\frac{1}{q}} & \text{if } 1 \le q < +\infty \\ \left\| t^{\frac{1}{p}} f^*(t) \right\|_{L^\infty(0,+\infty)} & \text{if } q = +\infty \end{cases}$$

is finite, where $f^* : [0, |\Omega|) \to \mathbb{R}$ denotes the nonincreasing rearrangement of $|f|$ such that

$$|\{x \in \Omega \mid |f(x)| \ge s\}| = |\{t \in [0, |\Omega|) \mid f^*(t) \ge s\}|, \ \forall s \ge 0.$$

For any $p > 1$, $L^{(p,q)}$ can be viewed as a deformation of L^p. In fact, there holds

$$L^{(p,p)} = L^p, \ L^{(p,1)} \subset L^{(p,q')} \subset L^{(p,q'')} \subset L^{(p,+\infty)} \text{ if } 1 < q' < q''.$$

Moreover, for any $p \in (1, +\infty)$ and $q \in [1, +\infty]$ we have $L^{(\frac{p}{p-1}, \frac{q}{q-1})}$ is the dual of $L^{(p,q)}$.

Now we have the following embedding theorem.

Theorem 3.2.7 *For $n \geq 2$, $W^{1,1}(\mathbb{R}^n)$ is continuously embedded in $L^{(\frac{n}{n-1},1)}(\mathbb{R}^n)$.*

Proof. See [198] for more details. Here we sketch the proof from [93] (page 142-143). Without loss of generality, assume $f \in C_0^\infty(\mathbb{R}^n)$. Let $\hat{f}$ be the nonincreasing rearrangement of $|f|$. Then we have

$$f^*\left(\alpha(n)r^n\right) = \hat{f}(r), \quad \alpha(n) = H^n(B_1).$$

Hence we have

$$\begin{aligned}
\int_{\mathbb{R}^n} |\nabla f| \;&\geq\; \int_{\mathbb{R}^n} |\nabla \hat{f}| \\
&= \int_0^{+\infty} \left(\int_{S^{n-1}} -\frac{\partial \hat{f}}{\partial r} r^{n-1}\, dH^{n-1} \right) dr \\
&= -H^{n-1}(S^{n-1}) \int_0^{+\infty} -\frac{\partial \hat{f}}{\partial r} r^{n-1}\, dr \\
&= (n-1)H^{n-1}(S^{n-1}) \int_0^{+\infty} \hat{f}(r) r^{n-2}\, dr \\
&= (n-1)\alpha(n)^{\frac{1}{n}} \int_0^{+\infty} t^{\frac{n-1}{n}} f^*(t) \frac{dt}{t} \\
&\geq C(n) \|f\|_{L^{(\frac{n}{n-1},1)}(\mathbb{R}^n)}.
\end{aligned}$$

This proves the lemma. □

In order to present the proof of Theorem 3.2.1 for $N = S^{L-1}$, we need one last theorem.

Theorem 3.2.8 *If $f \in H^1(\mathbb{R}^2)$ has compact support and $\nabla f \in L^{(2,1)}(\mathbb{R}^2)$, then $f \in C^0(\mathbb{R}^2)$.*

Proof. Note that $K(x) = \frac{1}{2\pi} \log \frac{1}{|x|}$ is the fundamental solution of the Laplacian equation in $\mathbb{R}^2$ i.e., $-\Delta K = \delta_0$ in $\mathbb{R}^2$ in $\mathcal{D}'(\mathbb{R}^2)$. Its derivative

$$\nabla K(x) = -\frac{1}{2\pi} \frac{x}{|x|^2}, \ x \in \mathbb{R}^2,$$

belongs to $L^{(2,\infty)}(\mathbb{R}^2)$.

Let's assume, in additions, $f \in C_0^\infty(\mathbb{R}^2)$. Then we have

$$\begin{aligned}
f(x) &= \int_{\mathbb{R}^2} \delta_0(x-y) f(y)\, dy \\
&= -\int_{\mathbb{R}^2} \Delta K(x-y) f(y)\, dy \\
&= -\int_{\mathbb{R}^2} \nabla K(x-y) \cdot \nabla f(y)\, dy
\end{aligned} \tag{3.22}$$

this implies that

$$\begin{aligned}\|f\|_{L^\infty(\mathbb{R}^2)} &\le \sup_{x\in\mathbb{R}^2} \|\nabla K(x-\cdot)\|_{L^{(2,\infty)}(\mathbb{R}^2)} \|\nabla f\|_{L^{(2,1)}(\mathbb{R}^2)} \\ &\le C \|\nabla f\|_{L^{(2,1)}(\mathbb{R}^2)} . \end{aligned} \tag{3.23}$$

For any $f \in H^1(\mathbb{R}^2)$ with $\mathrm{supp}(f)$ compact and $\nabla f \in L^{(2,1)}(\mathbb{R}^2)$, by the density theorem we have that there are $\{f_i\} \subset C_0^\infty(\mathbb{R}^2)$ with supports contained in a compact set such that

$$\lim_{i\to\infty} \left(\|f_i - f\|_{H^1(\mathbb{R}^2)} + \|\nabla(f_i - f)\|_{L^{(2,1)}(\mathbb{R}^2)} \right) = 0.$$

Hence applying (3.23) with f replaced by $f_i - f_j$ we have

$$\|f_i - f_j\|_{L^\infty(\mathbb{R}^2)} \le C\|\nabla(f_i - f_j)\|_{L^{(2,1)}(\mathbb{R}^2)},$$

so that $\{f_i\} \subset L^\infty(\mathbb{R}^2)$ is a Cauchy sequence. In particular, f is a uniform limit of a sequence of uniformly continuous functions in $\mathbb{R}^2$ and hence f is continuous. □

Proof of Theorem 3.2.1 for $N = S^{L-1}$:
Since the regularity is a local property, we assume that $M = B \subset \mathbb{R}^2$ is a unit ball. For $1 \le i, j \le L$, let $V^{ij} \in L^2(B, \mathbb{R}^2)$ be the divergene free vector fields given by lemma 3.2.2. Let $\hat{u} \in H^1(\mathbb{R}^2, \mathbb{R}^L)$ be an extension of u such that $\|\nabla \hat{u}\|_{L^2(\mathbb{R}^2)} \le C \|\nabla u\|_{L^2(B)}$, and $\hat{V}^{ij} \in L^2(\mathbb{R}^2, \mathbb{R}^2)$ be an extension of V^{ij} such that

$$\mathrm{div}\left(\hat{V}^{ij}\right) = 0, \ \left\|\hat{V}^{ij}\right\|_{L^2(\mathbb{R}^2)} \le C \left\|V^{ij}\right\|_{L^2(\mathbb{R}^2)} . \tag{3.24}$$

The existence of such $\hat{V}^{ij}$ can be obtained as follows. Since $\mathrm{div}(V^{ij}) = 0$ on B, there exists $\phi \in H^1(B)$ such that

$$V^{ij} = \left(\frac{\partial \phi}{\partial x_2}, -\frac{\partial \phi}{\partial x_1}\right) \text{ in } B.$$

Let $\hat{\phi} \in H^1(\mathbb{R}^2)$ be an extension of ϕ such that

$$\left\|\nabla\hat{\phi}\right\|_{L^2(\mathbb{R}^2)} \le C \|\nabla\phi\|_{L^2(B)} .$$

Then $\hat{V}^{ij} \equiv \left(\frac{\partial\hat{\phi}}{\partial x_2}, -\frac{\partial\hat{\phi}}{\partial x_1}\right)$ satisfies (3.24).

For $1 \le i \le L$, let $v^i \in L^1(\mathbb{R}^2)$ be a solution of

$$-\Delta v^i = \sum_{j=1}^{L} \nabla\hat{u}^j \cdot \hat{V}^{ij} \text{ in } \mathbb{R}^2. \tag{3.25}$$

By Theorem 3.2.5 and Theorem 3.2.4, we have that $\nabla^2 v^i \in L^1(\mathbb{R}^2)$ and

$$\begin{aligned}\left\|\nabla^2 v^i\right\|_{L^1(\mathbb{R}^2)} &\leq C\sum_{j=1}^{L}\left\|\nabla\hat{u}^j\cdot\hat{V}^{ij}\right\|_{\mathcal{H}^1(\mathbb{R}^2)}\\ &\leq C\sum_{j=1}^{L}\left\|\nabla\hat{u}^i\right\|_{L^2(\mathbb{R}^2)}\left\|\hat{V}^{ij}\right\|_{L^2(\mathbb{R}^2)}\\ &\leq C\sum_{j=1}^{L}\left\|\nabla u^j\right\|_{L^2(B)}\left\|V^{ij}\right\|_{L^2(B)}\leq C\left\|\nabla u\right\|^2_{L^2(B)}.\end{aligned}$$

Hence Theorem 3.2.7 implies that $\nabla v^i \in L^{(2,1)}(\mathbb{R}^2)$. Hence by Theorem 3.2.8 we have $v^i \in C^0_{\text{loc}}(\mathbb{R}^2)$.
Let $w^i = u^i - v^i : B \to \mathbb{R}$. Then it is readily seen that

$$\Delta w^i = 0 \text{ in } B$$

so that $w^i \in C^0(B)$. This implies that $u \in C^0(B, S^{L-1})$ and hence $u \in C^\infty(B, S^{L-1})$ by the higher order regularity theory for harmonic maps. □

It is clear that the symmetry of S^{L-1} plays an important role in the proof. For general target manifolds N without symmetry, Hélein [92] has developed a moving frame approach to the regularity issue of harmonic maps which we will present as follows.

Since a harmonic map $u : M \to N$ is invariant under a totally geodesic, isometrically embedding $\Phi : N \to \hat{N}$ (i.e., $\hat{u} = \Phi \circ u : M \to \hat{N}$ is harmonic), suitably choosing $\hat{N}$ and Φ can yield that the tangent bundle $T\hat{N}|_{\Phi(N)}$ is trivial, so that there exists an orthonormal tangent frame $\{\tilde{e}_\alpha\}$ of $\hat{N}$ on $\Phi(N)$. Hence $\hat{e}_\alpha = \tilde{e}_\alpha \circ \hat{u}$, $1 \leq \alpha \leq \dim(\hat{N})$, is a moving frame along $\hat{u}$. Because of this reduction, one can always assume that *there exists an orthonormal frame* $\{\tilde{e}_\alpha\}$, $1 \leq \alpha \leq k = dim(N)$, *of* u^*TN, *the pull back tangent bundle of* N, *such that*

$$\max_{1\leq\alpha\leq k}\left\|\nabla\tilde{e}_\alpha\right\|_{L^2(M)} \leq C\left\|\nabla u\right\|_{L^2(M)}. \tag{3.26}$$

By suitably rotating this frame, one then obtains a Coulomb gauge frame of u^*TN. More precisely, we have

Lemma 3.2.9 *There exists an orthonormal frame* $\{e_\alpha\}$, $1 \leq \alpha \leq k$, *of* u^*TN, *such that*

$$div\left(\langle\nabla e_\alpha, e_\beta\rangle\right) = 0 \quad in \quad M,\ 1 \leq \alpha, \beta \leq k, \tag{3.27}$$

and

$$\max_{1\leq\alpha\leq k}\int_M |\nabla e_\alpha|^2 \leq C\int_M |\nabla u|^2. \tag{3.28}$$

Proof. Denote $\mathrm{SO}(k) \subset \mathbb{R}^{k\times k}$ as the special orthogonal group of order k, and define

$$H^1(M, \mathrm{SO}(k)) = \left\{R \in H^1(M, \mathbb{R}^{k\times k}) \mid R^t(x)R(x) = \mathrm{I}_k,\ \det R = 1 \ \text{ a.e. } x\right\}.$$

For any $R \in H^1(M, \mathrm{SO}(k))$, let $e_\alpha \equiv \sum_{\beta=1}^k R_{\alpha\beta}\tilde{e}_\beta$, $1 \le \alpha \le k$, be another orthonormal frame of u^*TN, and define

$$E(R) = \sum_{\alpha,\beta=1}^{k} \int_M |\langle \nabla e_\alpha, e_\beta \rangle|^2 .$$

Now we consider the minimization problem:

$$\min\left\{E(R) \mid R \in H^1(M, \mathrm{SO}(k))\right\} \tag{3.29}$$

It was prove by [9] that $E(R)$ is sequentially lower semicontinuous with respect to weak convergence in $H^1(M, \mathrm{SO}(k))$ and there exists $R \in H^1(M, \mathrm{SO}(k))$ (see also [93] page 173-176) such that

$$E(R) = \min\left\{E(\hat{R}) \mid \hat{R} \in H^1(M, \mathrm{SO}(k))\right\}.$$

By a simple comparison, we have $E(R) \le E(I_k)$ so that

$$E(R) \le \sum_{\alpha=1}^{k} \int_M |\nabla \tilde{e}_\alpha|^2 \le C \int_M |\nabla u|^2.$$

Note that

$$\nabla e_\alpha = \sum_{\beta=1}^{k} \langle \nabla e_\alpha, e_\beta \rangle e_\beta + (\nabla e_\alpha)^\perp,$$

where $(\nabla e_\alpha)^\perp \in (T_uN)^\perp$ denotes the normal component of ∇e_α and can be estimated by

$$\left|(\nabla e_\alpha)^\perp\right| \le C|\nabla u|.$$

Hence we obtain (3.28). Direct computation of the first variational formula of $E(R)$ implies (3.27). □

Now we are ready to present the elegant proof of Theorem 3.2.1 by [92].
Proof of Theorem 3.2.1 for general N:
Assume $M = B^2 \subset \mathbb{R}^2$ and

$$\int_{B^2} |\nabla u|^2 \le \epsilon_0^2$$

for some small $\epsilon_0 > 0$ to be determined later. Let $\{e_\alpha\}$, $1 \le \alpha \le k$, be the Coulomb gauge frame of u^*TN obtained by Lemma 3.2.9. For $1 \le \alpha, \beta \le k$, denote $\omega_{\alpha\beta} = \langle \nabla e_\alpha, e_\beta \rangle$ as the connection of u^*TN. Since $\mathrm{div}(\omega_{\alpha\beta}) = 0$ in B^2, there is $\phi_{\alpha\beta} \in H^1(B^2)$ such that

$$\omega_{\alpha\beta} = \left(\frac{\partial \phi_{\alpha\beta}}{\partial x_2}, -\frac{\partial \phi_{\alpha\beta}}{\partial x_1}\right) \quad \text{in } B^2. \tag{3.30}$$

Therefore we have

$$\Delta \phi_{\alpha\beta} = \left\langle \frac{\partial e_\alpha}{\partial x_1}, \frac{\partial e_\alpha}{\partial x_2} \right\rangle - \left\langle \frac{\partial e_\alpha}{\partial x_2}, \frac{\partial e_\alpha}{\partial x_1} \right\rangle \quad \text{in } B^2. \tag{3.31}$$

Let $\bar{e}_\alpha$ be an extension of e_α to $\mathbb{R}^2$ such that

$$\|\nabla \bar{e}_\alpha\|_{L^2(\mathbb{R}^2)} \le C\,\|\nabla e_\alpha\|_{L^2(B^2)}.$$

Then by Theorem 3.2.5 we have that $\left\langle \frac{\partial \bar{e}_\alpha}{\partial x_1}, \frac{\partial \bar{e}_\alpha}{\partial x_2}\right\rangle - \left\langle \frac{\partial \bar{e}_\alpha}{\partial x_2}, \frac{\partial \bar{e}_\alpha}{\partial x_1}\right\rangle \in \mathcal{H}^1(\mathbb{R}^2)$. Hence if we let $\psi_{\alpha\beta} \in W^{2,1}(\mathbb{R}^2)$ solve

$$\Delta \psi_{\alpha\beta} = \left\langle \frac{\partial \bar{e}_\alpha}{\partial x_1}, \frac{\partial \bar{e}_\alpha}{\partial x_2}\right\rangle - \left\langle \frac{\partial \bar{e}_\alpha}{\partial x_2}, \frac{\partial \bar{e}_\alpha}{\partial x_1}\right\rangle,$$

then Theorems 3.2.4 and 3.2.7 imply that $\nabla\psi_{\alpha\beta} \in L^{(2,1)}(\mathbb{R}^2)$ so that we can assume that $\nabla\phi_{\alpha\beta} \in L^{(2,1)}(B^2)$ (since $\Delta(\phi_{\alpha\beta} - \psi_{\alpha\beta}) = 0$ in B^2) and hence $\omega_{\alpha\beta} \in L^{(2,1)}(B^2)$ with

$$\|\omega_{\alpha\beta}\|_{L^{(2,1)}(B^2)} \le C\,\|\nabla u\|_{L^2(B^2)} \le C\epsilon_0. \tag{3.32}$$

Now we use the complex notations. Let

$$z = x_1 + i x_2,\ \frac{\partial}{\partial z} = \frac{\partial}{\partial x_1} - i\frac{\partial}{\partial x_2},\ \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x_1} + i\frac{\partial}{\partial x_2},$$

and set

$$\mathcal{A}^\alpha = \left\langle \frac{\partial u}{\partial z}, e_\alpha \right\rangle,\ 1 \le \alpha \le k,$$

$$\omega_\alpha^\beta = \left\langle \frac{\partial e_\alpha}{\partial \bar{z}}, e_\beta \right\rangle,\ 1 \le \alpha, \beta \le k.$$

Then the harmonic map Equation (1.8) for u can be written as

$$\frac{\partial \mathcal{A}^\alpha}{\partial \bar{z}} = \sum_{\beta=1}^k \omega_\beta^\alpha \mathcal{A}^\beta, 1 \le \alpha \le k. \tag{3.33}$$

Putting

$$\mathcal{A} = \begin{pmatrix} \mathcal{A}^1 \\ \vdots \\ \mathcal{A}^k \end{pmatrix} \in \mathbb{C}^k,\ \omega = \begin{pmatrix} 0 & \omega_2^1 & \cdots & \omega_k^1 \\ \omega_1^2 & 0 & \cdots & \omega_k^2 \\ \vdots & \vdots & \ddots & \vdots \\ \omega_1^k & \omega_k^2 & \cdots & 0 \end{pmatrix} \in \mathrm{SO}(k) \otimes \mathbb{C},$$

we can write (3.33) as

$$\frac{\partial \mathcal{A}}{\partial \bar{z}} = \omega \mathcal{A} \text{ in } B^2. \tag{3.34}$$

Note that by (3.32) we have that $\omega \in L^{(2,1)}(B^2, \mathrm{SO}(k) \otimes \mathbb{C})$ and extend ω to $\mathbb{C}$ by taking the value 0 outside B^2. Then we have

$$\|\omega\|_{L^{(2,1)}(\mathbb{R}^2)} \le C\epsilon_0. \tag{3.35}$$

We now define a linear operator $T : L^\infty(\mathbb{C}, \mathbb{C}^{k\times k}) \to L^\infty(\mathbb{C}, \mathbb{C}^{k\times k})$ by

$$T(\mathcal{B})(z) \equiv \left((\frac{1}{\pi z}) * (\omega \mathcal{B}) \right)(z) = \int_{\mathbb{C}} \frac{\omega(w)\mathcal{B}(w)}{\pi(z-w)}.$$

Since $\frac{1}{\pi z} \in L^{(2,\infty)}(\mathbb{C})$, we can apply the duality between $L^{(2,1)}(\mathbb{C})$ and $L^{(2,\infty)}(\mathbb{C})$ to conclude that T maps $L^\infty(\mathbb{C}, \mathbb{C}^{k\times k})$ continuously to itself, with the operator bound

$$\|T\| \leq \left\| \frac{1}{\pi z} \right\|_{L^{(2,\infty)}(\mathbb{C})} \|\omega\|_{L^{(2,1)}(\mathbb{C})} \leq C\epsilon_0 < \frac{1}{2} \tag{3.36}$$

provided that ϵ_0 is chosen to be sufficiently small.

It is easy to see that $\frac{\partial}{\partial \bar{z}}\left(T(\mathcal{B})\right) = \omega\mathcal{B}$. Hence any solution $\mathcal{B}$ of (3.34) is also a solution of

$$\mathcal{B} - T(\mathcal{B}) = h, \tag{3.37}$$

where $h : \mathbb{C} \to \mathbb{C}^{k\times k}$ is a holomorphic function. Using a fixed point argument, we can conclude that for $h = \mathrm{I}_k$, the identity matrix of order k, (3.37) has a unique solution $\mathcal{B} \in L^\infty(\mathbb{C}, \mathbb{C}^{k\times k})$ such that

$$\|\mathcal{B} - \mathrm{I}_k\|_{L^\infty(\mathbb{C})} < \frac{1}{2}.$$

In particular, $\mathcal{B}$ takes values in the nonsingular matrices, $\mathrm{GL}(k, \mathbb{C})$. Furthermore, it is easy to see that $\mathcal{B}$ is also a solution of (3.34).

Let $\mathcal{B}^{-1}$ be the inverse matrix of $\mathcal{B}$. Then it follows from (3.33) and (3.34) that we have

$$\frac{\partial}{\partial \bar{z}}\left(\mathcal{B}^{-1}\mathcal{A}\right) = \frac{\partial \mathcal{B}^{-1}}{\partial \bar{z}}\mathcal{A} + \mathcal{B}^{-1}\frac{\partial \mathcal{A}}{\partial \bar{z}} = -\mathcal{B}^{-1}(\omega - \omega)\mathcal{A} = 0 \text{ in } B^2,$$

where we have used the fact that

$$\frac{\partial \mathcal{B}^{-1}}{\partial \bar{z}} = -\mathcal{B}^{-1}\frac{\partial \mathcal{B}}{\partial \bar{z}}\mathcal{B}^{-1} = -\mathcal{B}^{-1}\omega.$$

Hence $\mathcal{B}^{-1}\mathcal{A}$ is holomorphic in B^2 and is locally bounded. This implies that $\mathcal{A}$ is locally bounded in B^2, i.e. u is locally Lipschitz. Therefore $\Delta u \in L^\infty_{\mathrm{loc}}(B^2)$. By the L^p-theory and the Sobolev embedding theorem, we then have $u \in C^{1,\delta}(B^2)$ for some $0 < \delta < 1$. By virture of higher order regularity theory, we have $u \in C^\infty(B^2, N)$. □

Remark 3.2.10 For a Riemannian surface M with boundary ∂M, the boundary regularity of weakly harmonic maps under Dirichlet boundary data has been proved by Qing [156].

3.3 Stationary harmonic maps in higher dimensions

Weakly harmonic maps into manifolds whose sectional curvatures are either positve or changing signs can have wild behaviors in higher dimensions ($n \geq 3$). An example constructed by Riviére [160] shows that there exists a weakly harmonic map from B^3 to S^2 which is singular everywhere in B^3. On the other hand, there have been several important partial regularity theorems, due to Evans [45] and Bethuel [11], on the class of stationary harmonic maps. In this section, we will present these results.

Definition 3.3.1 A weakly harmonic map $u \in W^{1,2}(M,N)$ is *stationary* harmonic map, if it is a critical point of E with respect to the domain variations, i.e., for any family of differeomorphisms $F_t \in C^1(M \times [-1,1], M)$ with $F_0(x) = x$ for $x \in M$ and $F_t(x) = x$ for any $x \in \partial M$ and $t \in [-1,1]$, we have

$$\frac{d}{dt}|_{t=0} \int_M |\nabla (u \circ F_t)|_g^2 \, dv_g = 0. \tag{3.38}$$

Note that the differeomorphisms F_t take the form $F_t = e^{tX}$, where X is a smooth tangent vector field with compact support on M. Write $E(u = E_g(u)$ to indicate the role of the metric g. Then it is easy to see that, by a change of variables, we have

$$E_g(u \circ F_t) = E_{F_t^* g}(u),$$

where $F_t^* g$ is the pull-back of the metric g by F_t. By direct computation, we have

$$E_{(M,F_t * g)}(u) = E_{(M,g)}(u) + t \int_M \left(L_X g^{ij}\right) S_{ij}(u) \, dv_g + o(t), \tag{3.39}$$

where

$$S_{ij}(u) := \frac{1}{2} |\nabla u|^2 g_{ij} - \left\langle \frac{\partial u}{\partial x_i}, \frac{\partial u}{\partial x_j} \right\rangle,$$

is called the *stress-energy tensor*, and L_X is the Lie derivative in the direction of X. Since $(S_{ij}(u))$ is symmetric, we have

$$\left(2g^{ik} \nabla_{\frac{\partial}{\partial x_k}} X^j + L_X g^{ij}\right) S_{ij}(u) = 0.$$

Hence, by integration by parts, we obtain the following stationarity identity.

Proposition 3.3.2 *A map $u \in W^{1,2}(M,N)$ is stationary harmonic map iff $S_{ij}(u)$ is covariantly divergence free, i.e.,*

$$\nabla_i \left(g^{ik} S_{kj}(u)\right) = 0, \quad \forall 1 \leq j \leq n. \tag{3.40}$$

Remark 3.3.3 (i) If $M = \Omega \subset \mathbb{R}^n$ and g is the Euclidean metric, then, since the covariant derivative becomes the ordinary derivative, the stationarity identity (3.40 becomes a system of n conservation laws:

$$\sum_{1 \leq i \leq n} \frac{\partial S_{ij}(u)}{\partial x_i} = 0, \quad 1 \leq j \leq n$$

in the sense of distributions. Namely,

$$\int_\Omega \left\{ |\nabla u|^2 \mathrm{div}(Y) - 2 \langle u_i, u_j \rangle Y_i^j \right\} dx = 0, \quad \text{for all } Y \in C_0^1(\Omega, \mathbb{R}^n). \tag{3.41}$$

(ii) if $n = 2$ then $(S_{ij}(u))$ is trace free. Moreover, if $z = x_1 + ix_2$ is a local, conformal coordinate on M, then we can identify $S_{ij}(u)$ with the quadratic form $S := S_{ij}(u)\, dx_i\, dx_j$ so that

$$-2S = \mathrm{Re}\left\{\left(|\frac{\partial u}{\partial x_1}|^2 - |\frac{\partial u}{\partial x_2}|^2 - 2i\langle\frac{\partial u}{\partial x_1}, \frac{\partial u}{\partial x_2}\rangle\right)\right\} = 4\mathrm{Re}\mathcal{H}$$

where $\mathcal{H}$ is the Hopf differential of u given by

$$\mathcal{H} = h\left(\frac{\partial u}{\partial z}, \frac{\partial u}{\partial z}\right) dz^2.$$

In particular, u is a conformal map iff $S = 0$.

A very important property of stationary harmonic maps in dimension two is the following fact.

Proposition 3.3.4 *If $u \in W^{1,2}(M^2, N)$ is a stationary harmonic map, then its Hopf differential $\mathcal{H}$ is holomorphic and hence smooth.*

Proof. For simplicity, assume $(M^2, g) = (\Omega, g_0)$ with $\Omega \subset \mathbb{R}^2$ a bounded domain and g_0 the Euclidean metric. For $\phi \in C_0^1(\Omega)$, letting $Y(x, y) = (\phi(x, y), 0)$ and pluging it into (3.41), we obtain

$$\int_\Omega \left(\left(|u_y|^2 - |u_x|^2\right)\phi_x - 2\left\langle u_x, u_y\right\rangle \phi_y\right) dxdy = 0.$$

Similarly, choosing $Y(x, y) = (0, \psi(x, y))$ for $\psi \in C_0^1(\Omega)$, we get

$$\int_\Omega \left(\left(|u_x|^2 - |u_y|^2\right)\psi_y - 2\left\langle u_x, u_y\right\rangle \psi_x\right) dxdy = 0.$$

These two identities imply that $\left(|u_x|^2 - |u_y|^2\right) - 2i\left\langle u_x, u_y\right\rangle$ satisfies the Cauchy-Riemann equation in the distribution sense. Hence, by Weyl's lemma [145], $\mathcal{H}$ is holomorphic and hence smooth. □

Corollary 3.3.5 *If $u \in W^{1,2}(S^2, N)$ is a stationary harmonic map, then u is conformal, i.e., $|u_x|^2 - |u_y|^2 \equiv \langle u_x, u_y\rangle \equiv 0$ where (x, y) is a local conformal coordinate on S^2.*

Proof. Let $\Pi : S^2 \to \overline{\mathbb{R}^2}$ be the stereographic projection map, and $v = u \circ \Pi^{-1} : \mathbb{R}^2 \to N$. Then $v \in W^{1,2}(\mathbb{R}^2, N)$ is also a stationary harmonic map. Hence $\mathcal{H}(z) = (|v_x|^2 - |v_y|^2) - 2i\langle v_x, v_y\rangle \in L^1(\mathbb{R}^2)$ is holomorphic. By the Liouville theorem, we conclude $\mathcal{H} \equiv 0$. This implies that v, and hence u, is a conformal map. □

For $n \geq 3$, we have the following monotonicity formula for stationary harmonic maps derived by Price [155].

Proposition 3.3.6 *For $n \geq 3$ and $\Omega \subset \mathbb{R}^n$, if $u \in W^{1,2}(\Omega, N)$ is a stationary harmonic map, then for $x \in \Omega$ and $0 < r \leq R < \operatorname{dist}(x, \partial\Omega)$, it holds*

$$\begin{aligned} & R^{2-n} \int_{B_R(x)} |\nabla u|^2 - r^{2-n} \int_{B_r(x)} |\nabla u|^2 \\ & = 2 \int_{B_R(x) \setminus B_r(x)} |y-x|^{2-n} \left| \frac{\partial u}{\partial |y-x|} \right|^2 . \end{aligned} \tag{3.42}$$

Proof. Assume $x = 0$. Let $\eta_\epsilon = \eta_\epsilon(|x|) \in C_0^\infty(B_r)$ be such that $\eta_\epsilon = 1$ for $|x| \leq r(1-\epsilon)$. Let $Y(x) = \eta_\epsilon(x)x$, we have

$$Y_i^j(x) = \eta_\epsilon(|x|)\delta_{ij} + \eta_\epsilon'(|x|)\frac{x_i x_j}{|x|},$$

$$\operatorname{div} Y = n\eta_\epsilon(|x|) + \eta_\epsilon'(|x|)|x|.$$

Hence by (3.41), we have

$$\int_{B_r} |\nabla u|^2 \left((n-2)\eta_\epsilon(|x|) + 2\eta_\epsilon'(|x|)|x| \right) = 2 \int_{B_r} \eta_\epsilon'(|x|)|x| \left| \frac{\partial u}{\partial r} \right|^2 .$$

Sending ϵ to 0, this implies

$$(2-n) \int_{B_r} |\nabla u|^2 + r \int_{\partial B_r} |\nabla u|^2 = 2r \int_{\partial B_r} \left| \frac{\partial u}{\partial r} \right|^2 ,$$

or equivalently,

$$\frac{d}{dr} \left(r^{2-n} \int_{B_r} |\nabla u|^2 \right) = 2r^{2-n} \int_{\partial B_r} |\frac{\partial u}{\partial r}|^2.$$

Integrating it from r to R leads to (3.42). □

Remark 3.3.7 (i) It is clear that

$$\{ \text{ minimizing harmonic maps } \} \underset{\neq}{\subset} \{ \text{ stationary harmonic maps } \}.$$

By [19], if $g(z) = z^2 : S^2 \sim \overline{\mathbb{C}^2} \to S^2$ and $u(x) = g(\frac{x}{|x|}) : \mathbb{R}^3 \to S^2$, then u is a stationary, non-minimizing harmonic map.
(ii) Any C^2-harmonic map $u : M \to N$ is a stationary harmonic map.

Proof. For (ii), assume $M = \Omega \subset \mathbb{R}^n$. For $Y \in C_0^1(\Omega, \mathbb{R}^n)$, multiplying (1.8) by $Y \cdot \nabla u$ and integrating over Ω, we have by integration by parts

$$\begin{aligned} 0 &= \int_\Omega \langle \Delta u, Y \cdot \nabla u \rangle \, dx \\ &= \int_\Omega \left(\operatorname{div}(\langle \nabla u, Y \cdot \nabla u \rangle) - \langle u_i, u_j \rangle Y_i^j - \langle u_i, u_{ji} \rangle Y^j \right) dx \\ &= -\int_\Omega \langle u_i, u_j \rangle \, Y_i^j \, dx - \int_\Omega \left(\frac{|\nabla u|^2}{2} \right)_j Y^j \, dx \\ &= \frac{1}{2} \int_\Omega |\nabla u|^2 \operatorname{div} Y \, dx - \int_\Omega \langle u_i, u_j \rangle \, Y_i^j \, dx. \end{aligned}$$

This yields (3.41) and hence u is stationary harmonic map. □

For stationary harmonic maps, we have the following partial regularity theorem.

Theorem 3.3.8 *For $n \geq 3$, if $u \in W^{1,2}(M,N)$ is a stationary harmonic map, then $H^{n-2}(sing(u)) = 0$ and $u \in C^\infty(M \setminus sing(u), N)$.*

Remark 3.3.9 Theorem 3.3.8 was first proved by Evans [45] for $N = S^{L-1} \subseteq \mathbb{R}^L$ and then by Bethuel [11] for any Riemannian manifold $N \subseteq \mathbb{R}^L$. An alternative simpler proof of the main theorem of [45] was found by Chang-Wang-Yang [24]. Very recently, Riviére-Struwe [163] have improved [11].

In this section, we will present the proofs by [45] and [11]. First, introduce the Morrey space $M^{p,n-p}$ for $1 \leq p \leq n$.

Definition 3.3.10 For $1 \leq p \leq n$ and an open subset $U \subset \Omega$, a function $f \in L^p(U)$ belongs to the Morrey space, $M^{p,n-p}(U)$, if

$$\|f\|^p_{M^{p,n-p}(U)} := \sup_{B_r(x) \subset U} \left\{ r^{p-n} \int_{B_r(x)} |f|^p \right\} < +\infty.$$

It is clear that $M^{p,n-p}(U)$ equipped with the norm $\|\cdot\|_{M^{p,n-p}(U)}$ is a Banach space.

Note that $L^n(U) = M^{n,0}(U) \subset M^{p,n-p}(U)$ for any $1 \leq p < n$. On the other hand, from the point of view of scalings a function in $M^{p,n-p}(U)$ behaves exactly like a function in $L^n(U)$.

We also need to introduce BMO spaces.

Definition 3.3.11 For any open subset $U \subset \mathbb{R}^n$, a function $f \in L^1_{\text{loc}}(U)$ belongs to the BMO space, BMO(U), if

$$[f]_{\text{BMO}(U)} := \sup_{B_r(x) \subset U} \left\{ r^{-n} \int_{B_r(x)} |f - f_{x,r}| \right\} < +\infty,$$

where $f_{x,r} = \frac{1}{|B_r(x)|} \int_{B_r(x)} f(y)\, dy$ is the average of f over $B_r(x)$. It is clear that $[\cdot]_{\text{BMO}(U)}$ is a seminorm on BMO(U).

We need the duality between $\mathcal{H}^1(\mathbb{R}^n)$ and BMO$(\mathbb{R}^n)$, due to Fefferman-Stein [57].

Lemma 3.3.12 *Suppose $f \in H^1(\mathbb{R}^n)$, $g \in L^2(\mathbb{R}^n, \mathbb{R}^n)$ with $div(g) = 0$, and $h \in BMO(\mathbb{R}^n)$. Then $(\nabla f \cdot g)h$ is integrable in $\mathbb{R}^n$ and the following estimate holds*

$$\left| \int_{\mathbb{R}^n} (\nabla f \cdot g) h \right| \leq C \|\nabla f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)} [h]_{BMO(\mathbb{R}^n)}. \tag{3.43}$$

Proof. Since $g \in L^2(\mathbb{R}^n)$ is divergence free, it follows from Theorem 3.2.5 that $\nabla f \cdot g \in \mathcal{H}^1(\mathbb{R}^n)$ and

$$\|\nabla f \cdot g\|_{\mathcal{H}^1(\mathbb{R}^n)} \leq C \|\nabla f\|_{L^2(\mathbb{R}^n)} \|g\|_{L^2(\mathbb{R}^n)}. \tag{3.44}$$

By Fefferman-Stein's theorem [57], the dual space of $\mathcal{H}^1(\mathbb{R}^n)$ is $\mathrm{BMO}(\mathbb{R}^n)$. Hence (3.43) follows from (3.44). □

Note that if $u \in H^1(\Omega, N)$ is stationary harmonic map, then (3.42) implies that for any ball $B_{2r}(x) \subset \Omega$, $\nabla u \in M^{2,n-2}(B_r(x))$ and

$$\|\nabla u\|_{M^{2,n-2}(B_r(x))} \le C\left(r^{2-n}\int_{B_{2r}(x)} |\nabla u|^2\right)^{\frac{1}{2}}.$$

Hence, by the Poincaré inequality and Hölder inequality, we have

$$\begin{aligned} [u]_{\mathrm{BMO}(B_r(x))} &\le C\|\nabla u\|_{M^{1,n-1}(B_r(x))} \le C\|\nabla u\|_{M^{2,n-2}(B_r(x))} \\ &\le C\left(r^{2-n}\int_{B_{2r}(x)} |\nabla u|^2\right)^{\frac{1}{2}}. \end{aligned} \tag{3.45}$$

Proof of Theorem 3.3.8 for $N = S^{L-1}$:
By localization, iteration and application of Lemma 2.1.10, it suffices to show the following ϵ_0-decay lemma.

Lemma 3.3.13 *There are $\epsilon_0 = \epsilon_0(n, L) > 0$ and $\theta_0 \in (0, \frac{1}{2})$ such that if $u : B_2 \to S^{L-1}$ is a stationary harmonic map satisfying*

$$\int_{B_2} |\nabla u|^2\, dx \le \epsilon_0^2$$

then

$$\theta_0^{2-n}\int_{B_{\theta_0}} |\nabla u|^2 \le \frac{1}{2}\int_{B_1} |\nabla u|^2. \tag{3.46}$$

Proof. First (3.45) implies that

$$[u]_{\mathrm{BMO}(B_1)} \le C\left(\int_{B_2} |\nabla u|^2\right)^{\frac{1}{2}} \le C\epsilon_0.$$

Next by Lemma 3.2.2 we can write the harmonic map equation of u as

$$\Delta u^i = \sum_{1\le j\le L} \nabla u^j \cdot V^{ij} \ \text{ in } \ B_1,\ 1 \le i \le L, \tag{3.47}$$

for a vector field $V^{ij} = (V_1^{ij}, \cdots, V_n^{ij}) \in L^2(B_1, \mathbb{R}^n)$, with $\mathrm{div}(V^{ij}) = 0$, and $\|V^{ij}\|_{L^2(B_1)} \le C\|\nabla u\|_{L^2(B_1)}$.
Let $\hat{u}$ be an extension of u to $\mathbb{R}^n$ such that $\|\nabla \hat{u}\|_{L^2(\mathbb{R}^n)} \le C\|\nabla u\|_{L^2(B_1)}$, and $\hat{V}^{ij}$ be an extension of V^{ij} to $\mathbb{R}^n$ such that

$$\mathrm{div}\left(\hat{V}^{ij}\right) = 0 \text{ in } \mathbb{R}^n,\ \left\|\hat{V}^{ij}\right\|_{L^2(\mathbb{R}^n)} \le C\left\|V^{ij}\right\|_{L^2(B_1)}.$$

By Lemma 3.2.5 we have $\sum_{j=1}^{L} \nabla \hat{u}^j \cdot \hat{V}^{ij} \in \mathcal{H}^1(\mathbb{R}^n)$ for $1 \leq i \leq L$, and

$$\begin{aligned} \sum_{1 \leq j \leq L} \left\| \nabla \hat{u}^j \cdot \hat{V}^{ij} \right\|_{\mathcal{H}^1(\mathbb{R}^n)} &\leq C \sum_{1 \leq i,j \leq L} \|\nabla \hat{u}\|_{L^2(\mathbb{R}^n)} \left\| \hat{V}^{ij} \right\|_{L^2(\mathbb{R}^n)} \\ &\leq C \|\nabla u\|^2_{L^2(B_1)} \left(\leq C\epsilon_0^2\right) \end{aligned} \tag{3.48}$$

We now let $v : B_1 \to \mathbb{R}^L$ solve the Dirichlet problem

$$\begin{aligned} \Delta v &= 0, \quad \text{in } B_1 \\ v &= u, \quad \text{on } \partial B_1. \end{aligned}$$

By Lemma 3.2.5 and Lemma 3.3.12, we obtain

$$\begin{aligned} \int_{B_1} |\nabla(u-v)|^2 &= \sum_{1 \leq i,j \leq L} \int_{B_1} \nabla \hat{u}^j \cdot \hat{V}^{ij} \cdot (u-v)^i \\ &\leq C \sum_{1 \leq i,j \leq L} \left\| \nabla \hat{u}^j \cdot \hat{V}^{ij} \right\|_{\mathcal{H}^1(\mathbb{R}^n)} \|u-v\|_{\mathrm{BMO}(B_1)} \\ &\leq C\epsilon_0 \int_{B_1} |\nabla u|^2 \, dx \end{aligned}$$

where we have used the fact

$$[u-v]_{\mathrm{BMO}(B_1)} \leq 2\,[u]_{\mathrm{BMO}(B_1)} \leq C\epsilon_0.$$

Since v is harmonic, by the standard estimate we have that for any $\theta \in (0,1)$,

$$\begin{aligned} \theta^{2-n} \int_{B_\theta} |\nabla v|^2 \, dx &\leq \theta^2 \|\nabla v\|^2_{L^\infty(B_\theta)} \leq C\theta^2 \int_{B_1} |\nabla v|^2 \, dx \\ &\leq C\theta^2 \int_{B_1} |\nabla u|^2 \, dx. \end{aligned}$$

Combining these estimates together, we obtain

$$\theta^{2-n} \int_{B_\theta} |\nabla u|^2 \, dx \leq \left(C\theta^{2-n}\epsilon_0 + \theta^2\right) \int_{B_1} |\nabla u|^2 \, dx \leq \frac{1}{2} \int_{B_1} |\nabla u|^2 \, dx.$$

Therefore if we choose $\theta = \theta_0 \leq (4C)^{-\frac{1}{2}}$ and $\epsilon_0 \leq \frac{1}{4C}\theta_0^{n-2}$ then we obtain (3.46). □

Proof of Theorem 3.3.8 for general N:
By utilizing the Coulomb gauge frame construction developed by [92], Bethuel [11] has extended the argument by Evans [45] to show the partial regularity of stationary harmonic maps into target manifolds without symmetry. Here we will outline a slightly different proof of [11].

The crucial ingredient is the following decay lemma.

Lemma 3.3.14 *Under the same assumption of Theorem 3.3.8, there exist $\epsilon_0 > 0$ and $\theta_0 \in (0, \frac{1}{2})$ such that if $B_{2r}(x) \subset \Omega$ and*

$$(2r)^{2-n} \int_{B_{2r}(x)} |\nabla u|^2 \leq \epsilon_0^2, \tag{3.49}$$

then

$$\|\nabla u\|_{M^{1,n-1}(B_{\theta_0 r}(x))} \leq \frac{1}{2} \|\nabla u\|_{M^{1,n-1}(B_r(x))} . \tag{3.50}$$

We first need a Hodge decomposition of vector fields (cf. Iwaniec-Martin [101]).

Lemma 3.3.15 *For any* $1 < p < +\infty$ *and a ball* $B \subset \mathbb{R}^n$, *let* $F \in L^p(B, \mathbb{R}^n)$. *Then there exist* $G \in W_0^{1,p}(B)$ *and vector field* $H \in L^p(B, \mathbb{R}^n)$ *that is divergence free such that*

$$F = \nabla G + H \text{ in } B, \tag{3.51}$$

and

$$\|\nabla G\|_{L^p(B)} + \|H\|_{L^p(B)} \leq C\|F\|_{L^p(B)} \tag{3.52}$$

where $C > 0$ *depending only on* n *and* p.

Proof. It is standard that there exists a unique solution $G \in W_0^{1,p}(B)$ of

$$\Delta G = \operatorname{div}(F) \ \text{ in } B, \tag{3.53}$$

which satisfies the estimate

$$\|\nabla G\|_{L^p(B)} \leq C \|\nabla F\|_{L^p(B)} .$$

Now set $H = F - \nabla G$. It is easy to see that $\operatorname{div}(H) = 0$ in the distribution sense, and

$$\|H\|_{L^p(B)} \leq \left(\|F\|_{L^p(B)} + \|\nabla G\|_{L^p(B)}\right) \leq C\|\nabla F\|_{L^p(B)}.$$

This completes the proof. □

Next we need a dual characterization of $\nabla f \in L^p$.

Lemma 3.3.16 *For* $1 < p < +\infty$ *and a ball* $B \subset \mathbb{R}^n$, *let* $f \in W_0^{1,p}(B)$. *Then there exists* $C > 0$ *depending on* n *and* p *such that*

$$\|\nabla f\|_{L^p(B)} \leq \sup \left\{ \int_B \nabla f \cdot \nabla h \mid h \in W_0^{1,p'}(B), \ \|\nabla h\|_{L^{p'}(B)} \leq C \right\} \tag{3.54}$$

where $p' = \frac{p}{p-1}$.

Proof. Note first that there exists $F \in L^{p'}(B, \mathbb{R}^n)$, with $\|F\|_{L^{p'}(B)} = 1$, such that

$$\|\nabla f\|_{L^p(B)} = \int_B \nabla f \cdot F.$$

Applying Lemma 3.3.15 yields that there are $G \in W_0^{1,p'}(B)$ and $H \in L^{p'}(B, \mathbb{R}^n)$ with $\operatorname{div}(H) = 0$ such that

$$F = \nabla G + H \ \text{ in } \ B,$$

$$\|\nabla G\|_{L^{p'}(B)} \leq C \|F\|_{L^{p'}(B)} = C.$$

Hence, by integration by parts and using $\mathrm{div}(H) = 0$, we have

$$\begin{aligned}\int_B \nabla f \cdot F &= \int_B \nabla f \cdot (\nabla G + H) \\ &= \int_B \nabla f \cdot \nabla G.\end{aligned}$$

This gives (3.54) and completes the proof . □

Proof of Lemma 3.3.14:
For simplicity, assume $x = 0$, $r = 1$ and $B_2 \subset \Omega$. For any ball $B_\rho(z) \subset B_1$, by Lemma 3.2.9 there is a Coulomb gauge frame $\{e_\alpha\}$, $1 \le \alpha \le l$, of u^*TN over $B_\rho(z)$ such that

$$\sum_{\alpha=1}^{l} \|\nabla e_\alpha\|_{L^2(B_\rho(z))} \le C\|\nabla u\|_{L^2(B_\rho(z))}. \tag{3.55}$$

Let $\eta \in C_0^\infty(\mathbb{R}^n)$ be such that $0 \le \eta \le 1$, $\eta \equiv 1$ on $B_{\frac{3\rho}{4}}(z)$, $\eta = 0$ outside $B_\rho(z)$, and $|\nabla\eta| \le 8\rho^{-1}$. Denote $u_{z,\rho}$ as the average of u over $B_\rho(z)$. For $1 \le \alpha \le l$, consider the vector field $\phi^\alpha = \langle \nabla(\eta(u - u_{z,\rho})), e_\alpha\rangle : B_\rho(z) \to \mathbb{R}^n$. Note that by the Poincaré inequality we have

$$\|\phi^\alpha\|_{L^2(B_\rho(z))} \le \|\nabla u\|_{L^2(B_\rho(z)}. \tag{3.56}$$

Applying Lemma 3.3.15, we have that $g^\alpha \in H_0^1(B_\rho(z))$ and $h^\alpha \in L^2(B_\rho(z), \mathbb{R}^n)$ such that $\mathrm{div}(h^\alpha) = 0$ in $B_\rho(z)$,

$$\phi^\alpha = \nabla g^\alpha + h^\alpha \text{ in } B_\rho(z), \tag{3.57}$$

and

$$\|\nabla g^\alpha\|_{L^2(B_\rho(z))} + \|h^\alpha\|_{L^2(B_\rho(z))} \le C\|\phi^\alpha\|_{L^2(B_\rho(z))} \le C\|\nabla u\|_{L^2(B_\rho(z))}. \tag{3.58}$$

Taking divergence of both sides of (3.57), we have

$$\Delta g^\alpha = \mathrm{div}(\phi^\alpha) \text{ in } B_\rho(z),$$

which implies that in $B_{\frac{\rho}{2}}(z)$ (using the fact that u is weakly harmonic)

$$\begin{aligned}\Delta g^\alpha &= \mathrm{div}\left(\langle \nabla u, e_\alpha\rangle\right) = \langle \nabla u, \nabla e_\alpha\rangle \\ &= \sum_{\beta=1}^{l} \langle \nabla u \langle \nabla e_\alpha, e_\beta\rangle, e_\beta\rangle .\end{aligned} \tag{3.59}$$

Decompose g^α in $B_{\frac{\rho}{2}}(z)$ as follows:

$$g^\alpha = g_1^\alpha + g_2^\alpha \text{ in } B_{\frac{\rho}{2}}(z), \tag{3.60}$$

and

$$\Delta g_2^\alpha = 0 \text{ in } B_{\frac{\rho}{2}}(z),\ g_2^\alpha = g^\alpha \text{ on } \partial B_{\frac{\rho}{2}}(z). \tag{3.61}$$

We estimate g_i^α $(i = 1, 2)$ and h^α as follows.

Step 1. Estimate h^α:
Using (3.57) and $\mathrm{div}(h^\alpha)=0$ and integration by parts, we have

$$\begin{aligned}\int_{B_\rho(z)}|h^\alpha|^2 &= \int_{B_\rho(z)} h^\alpha\cdot(\phi^\alpha-\nabla g^\alpha)=\int_{B_\rho(z)} h^\alpha\cdot\phi^\alpha\\ &= \int_{B_\rho(z)}\langle\nabla(\eta(u-u_{z,\rho})),e_\alpha\rangle\cdot h^\alpha\\ &= -\int_{B_\rho(z)}\langle\eta(u-u_{z,\rho}),\nabla e_\alpha\rangle\cdot h^\alpha.\end{aligned}$$

Let $\hat{e}_\alpha$ be an extension of e_α to $\mathbb{R}^n$ such that

$$\|\nabla\hat{e}_\alpha\|_{L^2(\mathbb{R}^n)}\le C\|\nabla e_\alpha\|_{L^2(B_{\frac{\rho}{2}}(z))},$$

and $\hat{h}^\alpha$ be an extension of h^α to $\mathbb{R}^n$ such that

$$\mathrm{div}(\hat{h}^\alpha)=0\ \text{ in }\ \mathbb{R}^n,\ \|\hat{h}^\alpha\|_{L^2(\mathbb{R}^n)}\le C\|h^\alpha\|_{L^2(B_{\frac{\rho}{2}}(z))}.$$

Then we have

$$\int_{B_{\frac{\rho}{2}}(z)}|h^\alpha|^2=-\sum_{\beta=1}^{l}\int_{\mathbb{R}^n}\langle\eta(u-u_{z,\rho}),\nabla\hat{e}_\alpha\rangle\cdot\hat{h}^\alpha.$$

Since $\nabla\hat{e}_\alpha\cdot\hat{h}^\alpha\in\mathcal{H}^1(\mathbb{R}^n)$ and $\eta(u-u_{z,\rho})\in\mathrm{BMO}(\mathbb{R}^n)$, we have

$$\begin{aligned}\int_{B_{\frac{\rho}{2}}(z)}|h^\alpha|^2 &\le C\left\|\nabla\hat{e}_\alpha\cdot\hat{h}^\alpha\right\|_{\mathcal{H}^1(\mathbb{R}^n)}[\eta(u-u_{z,\rho})]_{\mathrm{BMO}(\mathbb{R}^n)}\\ &\le C\|\nabla\hat{e}_\alpha\|_{L^2(\mathbb{R}^n)}\left\|\hat{h}^\alpha\right\|_{L^2(\mathbb{R}^n)}[\eta(u-u_{z,\rho})]_{\mathrm{BMO}(\mathbb{R}^n)}\\ &\le C\|\nabla e_\alpha\|_{L^2(B_{\frac{\rho}{2}}(z))}\|h^\alpha\|_{L^2(B_{\frac{\rho}{2}}(z))}[\eta(u-u_{z,\rho})]_{\mathrm{BMO}(\mathbb{R}^n)}.\end{aligned}$$

Now we claim that

$$[\eta(u-u_{z,\rho})]_{\mathrm{BMO}(\mathbb{R}^n)}\le C\|\nabla u\|_{M^{1,n-1}(B_1)}. \tag{3.62}$$

To prove (3.62), let $z_0\in B_{\frac{3\rho}{4}}(z)$ and $0<r\le\frac{\rho}{4}$ so that $B_r(z_0)\subset B_\rho(z)$. We first have

$$\begin{aligned}\sup_{B_r(z_0)}\left|(\eta(u-u_{z,\rho}))_{z_0,r}-\eta\left(u-u_{z,\rho}\right)_{z_0,r}\right| &\le \sup_{B_r(z_0)}|\eta(x)-\eta(y)|\,|(u-u_{z,\rho})_{z_0,r}|\\ &\le C\frac{r}{\rho}\left(\frac{1}{r^n}\int_{B_r(z_0)}|u-u_{z,\rho}|\right)\\ &\le C\rho^{-1}\left(\int_{B_r(z_0)}|u-u_{z,\rho}|^n\right)^{\frac{1}{n}}\\ &\le C\left(\rho^{-n}\int_{B_\rho(z)}|u-u_{z,\rho}|^n\right)^{\frac{1}{n}}\\ &\le C[u]_{\mathrm{BMO}(B_1)}\le C\|\nabla u\|_{M^{1,n-1}(B_1)},\end{aligned}$$

where we have used the Hölder inequality in the last two steps and John-Nirenberg inequality [109] (using the fact that $u \in \mathrm{BMO}(B_1)$), namely

$$\left(\rho^{-n}\int_{B_\rho(z)}|u-u_{z,\rho}|^n\right)^{\frac{1}{n}} \le C\rho^{-n}\int_{B_\rho(z)}|u-u_{z,\rho}| \le C\,[u]_{\mathrm{BMO}(B_1)}$$

in the last step.
Therefore we obtain, for $z_0 \in B_{\frac{3\rho}{4}}(z)$ and $0 < r \le \frac{\rho}{4}$,

$$\begin{aligned}
&r^{-n}\int_{B_r(z_0)}|\eta(u-u_{z,\rho})-(\eta(u-u_{z,\rho}))_{z_0,r}|\\
&\le r^{-n}\int_{B_r(z_0)}\eta\,|u-u_{z_0,r}| + \sup_{B_r(z_0)}|(\eta(u-u_{z,\rho}))_{z_0,r}-\eta(u-u_{z,\rho})_{z_0,r}|\\
&\le Cr^{-n}\int_{B_r(z_0)}|u-u_{z_0,r}| + C\,\|\nabla u\|_{M^{1,n-1}(B_1)}\\
&\le C\,\|\nabla u\|_{M^{1,n-1}(B_1)}\,.
\end{aligned}$$

A similar inequality holds trivially for either $z_0 \in B_{\frac{3\rho}{4}}(z)$ and $r > \frac{\rho}{4}$, or $z_0 \in \mathbb{R}^n \setminus B_{\frac{3\rho}{4}}(z)$ (since $\eta = 0$ outside $B_{\frac{3\rho}{4}}(z)$). Hence (3.62) follows.
Applying (3.62) and using the Hölder inequality, we obtain

$$\begin{aligned}
\rho^{1-n}\int_{B_{\frac{\rho}{2}}(z)}|h^\alpha| &\le \left(\rho^{2-n}\int_{B_{\frac{\rho}{2}}(z)}|h^\alpha|^2\right)^{\frac{1}{2}}\\
&\le C\left(\rho^{2-n}\int_{B_\rho(z)}|\nabla u|^2\right)^{\frac{1}{2}}\|\nabla u\|_{M^{1,n-1}(B_1)}\\
&\le C\epsilon_0\|\nabla u\|_{M^{1,n-1}(B_1)}. \qquad (3.63)
\end{aligned}$$

Step 2. Estimate ∇g_1^α
Note that

$$\begin{aligned}
\Delta g_1^\alpha &= \sum_{\beta=1}^k \langle \nabla u\langle e_\alpha, e_\beta\rangle, e_\beta\rangle \text{ in } B_{\frac{\rho}{2}}(z) \qquad (3.64)\\
g_1^\alpha &= 0 \text{ on } \partial B_{\frac{\rho}{2}}(z).
\end{aligned}$$

By Lemma 3.54, we have that for $1 < p < \frac{n}{n-1}$ there is $\xi \in W_0^{1,p'}(B_{\frac{\rho}{2}}(z))$ with $\|\nabla \xi\|_{W_0^{1,p'}(B_{\frac{\rho}{2}}(z))} \le C$ such that

$$\|\nabla g_1^\alpha\|_{L^p(B_{\frac{\rho}{2}}(z))} = \int_{B_{\frac{\rho}{2}}(z)} \nabla g_1^\alpha \cdot \nabla \xi.$$

Multiplying (3.64) by ξ and using integration by parts and $\operatorname{div}(\langle\nabla e_\alpha, e_\beta\rangle) = 0$, we have

$$\|\nabla g_1^\alpha\|_{L^p(B_{\frac{\rho}{2}}(z))} = -\sum_{\beta=1}^k \int_{B_{\frac{\rho}{2}}(z)} u\,\langle\nabla e_\alpha, e_\beta\rangle\cdot\nabla(\xi e_\beta).$$

Let $\hat{u}$ be an extension of u to $\mathbb{R}^n$ such that

$$[\hat{u}]_{\mathrm{BMO}(\mathbb{R}^n)} \le C\,[u]_{\mathrm{BMO}(B_{\frac{\rho}{2}}(z))}\left(\le C\|\nabla u\|_{M^{1,n-1}(B_1)}\right),$$

and $\omega_{\alpha\beta}$ be an extension of $\langle\nabla e_\alpha, e_\beta\rangle$ to $\mathbb{R}^n$ such that

$$\operatorname{div}(\omega_{\alpha\beta}) = 0 \ \text{ in } \ \mathbb{R}^n,$$

$$\|\omega_{\alpha\beta}\|_{L^2(\mathbb{R}^n)} \le C\|\langle\nabla e_\alpha, e_\beta\rangle\|_{L^2(B_{\frac{\rho}{2}}(z))} \le C\|\nabla u\|_{L^2(B_\rho(z))}.$$

Then we have

$$\begin{aligned}
\|\nabla g_1^\alpha\|_{L^p(B_{\frac{\rho}{2}}(z))} &= -\sum_{\beta=1}^k \int_{\mathbb{R}^n}\langle\hat{u}, \omega_{\alpha\beta}\rangle\cdot\nabla(\xi e_\beta)\\
&\le C\sum_{\beta=1}^k \|\omega_{\alpha\beta}\cdot\nabla(\xi e_\beta)\|_{\mathcal{H}^1(\mathbb{R}^n)}[\hat{u}]_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\le C\sum_{\beta=1}^k \|\omega_{\alpha\beta}\|_{L^2(\mathbb{R}^n)}\|\nabla(\xi e_\beta)\|_{L^2(\mathbb{R}^n)}[\hat{u}]_{\mathrm{BMO}(\mathbb{R}^n)}\\
&\le C\sum_{\beta=1}^k [u]_{\mathrm{BMO}(B_{\frac{\rho}{2}}(z))}\|\nabla u\|_{L^2(B_\rho(z))}\|\nabla(\xi e_\beta)\|_{L^2(\mathbb{R}^n)}. \quad (3.65)
\end{aligned}$$

Note that since $1 < p < \frac{n}{n-1}$, we have $p' = \frac{p}{p-1} > n$. Hence by the Sobolev embedding theorem, we have that $\xi \in W_0^{1,p'}\left(B_{\frac{\rho}{2}}(z)\right) \subset C^{1-\frac{n}{p'}}\left(B_{\frac{\rho}{2}}(z)\right)$ and

$$\|\xi\|_{L^\infty(B_{\frac{\rho}{2}}(z))} \le C\rho^{1-\frac{n}{p'}}\|\nabla\xi\|_{L^{p'}(B_{\frac{\rho}{2}}(z))} \le C\rho^{1-\frac{n}{p'}}.$$

Therefore by the Hölder inequality, we have

$$\begin{aligned}
\|\nabla(\xi e_\alpha)\|_{L^2(\mathbb{R}^n)} &\le \|\nabla\xi\|_{L^2(B_{\frac{\rho}{2}}(z))} + \|\nabla e_\beta\|_{L^2(B_{\frac{\rho}{2}}(z))}\|\xi\|_{L^\infty(B_{\frac{\rho}{2}}(z))}\\
&\le C\left(\rho^{\frac{n}{2}-\frac{n}{p'}}\|\nabla\xi\|_{L^{p'}(B_{\frac{\rho}{2}}(z))} + \rho^{1-\frac{n}{p'}}\|\nabla u\|_{L^2(B_\rho(z))}\right)\\
&\le C\left(1+\|\nabla u\|_{M^{2,n-2}(B_1)}\right)\rho^{\frac{n}{2}-\frac{n}{p'}} \le C\rho^{\frac{n}{2}-\frac{n}{p'}}. \quad (3.66)
\end{aligned}$$

Putting (3.66) into (3.65) and using the Hölder inequality, we obtain

$$\begin{aligned}
\rho^{1-n}\int_{B_{\frac{\rho}{2}}(z)}|\nabla g_1^\alpha| &\le \rho^{1-\frac{n}{p}}\,\|\nabla g_1^\alpha\|_{L^p(B_{\frac{\rho}{2}}(z))}\\
&\le C\rho^{(1-\frac{n}{p}+\frac{n}{2}-\frac{n}{p'})}\,\|\nabla u\|_{L^2(B_\rho(z))}\,[u]_{\mathrm{BMO}(B_{\frac{\rho}{2}}(z))}\\
&\le C\rho^{(1-\frac{n}{p}+\frac{n}{2}-\frac{n}{p'}+\frac{n-2}{2})}\epsilon_0\,[u]_{\mathrm{BMO}(B_{\frac{\rho}{2}}(z))}\\
&\le C\epsilon_0\,[u]_{\mathrm{BMO}(B_{\frac{\rho}{2}}(z))}\\
&\le C\epsilon_0\,\|\nabla u\|_{M^{1,n-1}(B_1)}\,. \qquad (3.67)
\end{aligned}$$

Step 3. Estimate ∇g_2^α

Since $\Delta g_2^\alpha = 0$ in $B_{\frac{\rho}{2}}(z)$, by the standard estimate on harmonic functions we have that for any $\theta \in (0, \frac{1}{2})$,

$$\begin{aligned}
(\theta\rho)^{1-n}\int_{B_{\theta\rho}(z)}|\nabla g_2^\alpha| &\le C\theta\rho^{1-n}\int_{B_{\frac{\rho}{2}}(z)}|\nabla g_2^\alpha|\\
&\le C\theta\rho^{-n}\int_{B_{\frac{\rho}{2}}(z)}\left(|\nabla\phi^\alpha|+|\nabla g_1^\alpha|+|h^\alpha|\right)\\
&\le C\theta\rho^{-n}\int_{B_\rho(z)}|\nabla\left(\eta(u-u_{z,\rho})\right)|+C\epsilon_0\,\|\nabla u\|_{M^{1,n-1}(B_1)}\\
&\le C\left(\theta+\epsilon_0\right)\|\nabla u\|_{M^{1,n-1}(B_1)} \qquad (3.68)
\end{aligned}$$

where we have used the Poincaré inequality, (3.67) and (3.68) in the last two inequalities.

Putting (3.63), (3.67) and (3.68) together gives that for any $B_\rho(z) \subset B_1$ and $\theta \in (0, \frac{1}{2})$,

$$(\theta\rho)^{1-n}\int_{B_{\theta\rho}(z)}|\nabla u| \le C\left(\theta+\epsilon_0+\theta^{1-n}\epsilon_0\right)\|\nabla u\|_{M^{1,n-1}(B_1)}\,. \qquad (3.69)$$

Taking supremum of (3.69) over all $B_\rho(z) \subset B_1$, we have

$$\|\nabla u\|_{M^{1,n-1}(B_\theta)} \le C\left(\theta+\epsilon_0+\theta^{1-n}\epsilon_0\right)\|\nabla u\|_{M^{1,n-1}(B_1)}\,. \qquad (3.70)$$

Therefore if we first choose $\theta_0 = \theta \in (0, \frac{1}{2})$ sufficiently small and then choose ϵ_0 sufficiently small to ensure $C\left(\theta+\epsilon_0+\theta^{1-n}\epsilon_0\right) < \frac{1}{2}$, then (3.70) yields (3.50). The proof of Lemma 3.3.14 is complete. □

Proof of Theorem 3.3.8:

Let $x \in \Omega$ and choose $R > 0$ such that $B_{4R}(x) \subset \Omega$ and

$$(4R)^{2-n}\int_{B_{4R}(x)}|\nabla u|^2 \le \epsilon_0^2.$$

Then an elementary calculation yields that

$$(2R)^{2-n}\int_{B_{2R}(y)}|\nabla u|^2\le 2^{n-2}\epsilon_0^2,\ \forall y\in B_{2R}(x).$$

Hence (3.42) implies that for any $r\in(0,R)$,

$$r^{2-n}\int_{B_{2r}(y)}|\nabla u|^2\le 2^{n-2}\epsilon_0^2,\ \forall y\in B_{2R}(x).$$

Applying Lemma 3.3.14 we deduce that there is $\theta_0\in(0,\frac{1}{2})$ such that

$$\|\nabla u\|_{M^{1,n-1}(B_{\theta r}(y))}\le\frac{1}{2}\|\nabla u\|_{M^{1,n-1}(B_r(y))},\ \forall y\in B_{2R}(x).$$

Hence by finitely many iterations we obtain that for any $y\in B_{2R}(x)$ and $0<r\le R$,

$$\|\nabla u\|_{M^{1,n-1}(B_r(y))}\le C\left(\frac{r}{R}\right)^{\alpha_0}\|\nabla u\|_{M^{1,n-1}(B_R(y))}, \tag{3.71}$$

where $\alpha_0=\frac{\log 2}{|\log\theta_0|}\in(0,1)$. This, combined with the Poincaré inequality, implies that for any $y\in B_{2R}(x)$ and $0<r\le R$, we have

$$\frac{1}{r^n}\int_{B_r(y)}|u-u_{y,r}|\le Cr^{\alpha_0}$$

so that $u\in C^{\alpha_0}(B_{2R}(x),N)$ and by the higher order regularity $u\in C^\infty(B_{2R}(x),N)$.
□

Remark 3.3.17 When $M\neq\emptyset$, we can consider the Dirichlet boundary value problem of stationary harmonic maps with smooth boundary data $\phi\in C^\infty(\partial\Omega,N)$. Wang [205] has proved a boundary partial regularity theorem: Under the additional assumption that the corresponding boundary monotonicity inequality holds: there exist $r_0=\rho_0(\partial M,\phi)>0$ and $C_0=C_0(\partial M,\phi)>0$ such that for any $a\in\partial M$ and $0<r\le R\le r_0$,

$$r^{2-n}\int_{M\cap B_r(a)}|\nabla u|^2\le e^{C_0R}R^{2-n}\int_{M\cap B_R(a)}|\nabla u|^2+C_0(R-r). \tag{3.72}$$

Then there exists a closed subset $\Sigma\subset\overline{M}$, with $H^{n-2}(\Sigma)=0$, such that $u\in C^\infty(\overline{M}\setminus\Sigma,N)$.

3.4 Stable-stationary harmonic maps into spheres

In this section, we will discuss the class of stable-stationary harmonic maps that lie between minimizing harmonic maps and stationary harmonic maps. The notion is motived by that of minimal hyersurfaces by Schoen-Simon [170]. The theorems we will present here extend [174] on minimizing harmonic maps into spheres. We only consider $N=S^k\subseteq\mathbb{R}^{k+1}$, $k\ge 3$.

Definition 3.4.1 For $k \geq 3$, a stationary harmonic map $u \in W^{1,2}(\Omega, S^k)$ is stable, if in addition

$$\frac{d^2}{dt^2}|_{t=0} \int_\Omega \left|\nabla\left(\frac{u+t\phi}{|u+t\phi|}\right)\right|^2 \geq 0 \text{ for all } \phi \in C_0^1(\Omega, \mathbb{R}^{k+1}). \tag{3.73}$$

It follows from §1.6 and [174] that (3.73) gives

$$\int_\Omega |\nabla\eta|^2 \geq \frac{k-2}{k}\int_\Omega |\nabla u|^2\,\eta^2, \text{ for all } \eta \in C_0^1(\Omega). \tag{3.74}$$

For stable-stationary harmonic maps, Hong-Wang [98] have proved the following theorem.

Theorem 3.4.2 *For $k \geq 3$, if $u \in W^{1,2}(\Omega, S^k)$ is a stable-stationary harmonic map, then $dim_H(sing(u)) \leq n - d(k) - 1$, where*

$$d(k) = \begin{cases} 3, & k = 3 \\ \min\left\{\left[\frac{k}{2}\right]+1, 6\right\}, & k \geq 4 \end{cases}$$

and $[t]$ denotes the largest integer part of t.

Remark 3.4.3 For $k \geq 2$, Schoen-Uhlenbeck [174] have proved Theorem 3.4.2 for any minimizing harmonic maps. Since (3.74) is void for $k = 2$, it is unknown whether Theorem 3.4.2 holds for stable-stationary harmonic maps into S^2.

The crucial step to establish Theorem 3.4.2 is to establish the sequential compactness among a family of stable-stationary harmonic maps. This is done by utlizing some potential theory. More precisely, we have

Lemma 3.4.4 *For any $A > 0$, let $\mathcal{C}_A$ consist of stationary harmonic maps $u \in W^{1,2}(\Omega, N)$ satisfying*

$$\int_\Omega |\nabla\eta|^2 \geq A\int_\Omega |\nabla u|^2\eta^2 \text{ for all } \eta \in C_0^\infty(\Omega). \tag{3.75}$$

Then $\mathcal{C}_A$ is sequentially compact in $W^{1,2}(\Omega, N)$.

Proof. Note first that (3.75) implies that $\mathcal{C}_A \subset W^{1,2}(\Omega, N)$ is locally uniformly bounded. Hence for any $\{u_i\} \subset \mathcal{C}_A$, we may assume that there is $u \in W^{1,2}(\Omega, N)$ such that $u_i \to u$ weakly in $W^{1,2}(\Omega, N)$ as $i \to \infty$. Define the concentration set

$$\Sigma = \bigcap_{r>0}\left\{x \in \Omega : \liminf_{i\to\infty} r^{2-n}\int_{B_r(x)} |\nabla u_i|^2 \geq \epsilon_0^2\right\},$$

where $\epsilon_0 > 0$ is given by Lemma 3.3.14. Since u_i satisfies (3.42), a standard covering argument (see [166]) implies that Σ is closed and

$$H^{n-2}(\Sigma \cap K) < \infty, \quad \forall K \subset\subset \Omega.$$

Moreover, by Lemma 3.3.14, we can assume that

$$u_i \to u \text{ in } C^2_{\text{loc}} \cap W^{1,2}_{\text{loc}} (\Omega \setminus \Sigma, N).$$

We claim that *if* $H^{n-2}(\Sigma) = 0$, *then* $u_i \to u$ *strongly in* $W^{1,2}(\Omega, N)$. To prove it, we proceed as follows. Note that there exists a nonnegative Radon measure ν on Ω such that

$$|\nabla u_i|^2 \, dx \to |\nabla u|^2 \, dx + \nu, \quad \text{as } i \to \infty,$$

as convergence of Radon measures.

Denote $\mu = |\nabla u|^2 \, dx + \nu$. Then, by (3.42), we have

$$r^{2-n} \mu (B_r(x)) \le R^{2-n} \mu (B_R(x))$$

for any $x \in \Omega$, $0 < r \le R < \text{dist}(x, \partial\Omega)$. Hence

$$\Theta^{n-2}(\mu, x) = \lim_{r \to 0} r^{2-n} \mu (B_r(x))$$

exists and is upper semicontinuous for any $x \in \Omega$. Since $\nu(\Omega \setminus \Sigma) = 0$, $\text{supp}(\nu) \subseteq \Sigma$. It follows from the definition of Σ that

$$\epsilon_0^2 \le \Theta^{n-2} (\mu, a), \ \forall a \in \Sigma.$$

On the other hand, we have, for any compact $K \subset\subset \Omega$,

$$\Theta^{n-2} (\mu, a) \le C (K, E), \ \forall a \in \Sigma \cap K,$$

where $E = \sup_i \int_\Omega |\nabla u_i|^2 < \infty$. In fact, for $a \in \Sigma \cap K$ and $r_0 = \frac{1}{2}\text{dist}(a, \partial\Omega)$, we have

$$\Theta^{n-2}(\mu, a) \le r_0^{2-n} \mu (B_{r_0}(a)) = r_0^{2-n} \lim_{i \to \infty} \int_{B_{r_0}(a)} |\nabla u_i|^2 \le r_0^{2-n} E.$$

Now we have (cf. [48]) that for any compact $K \subset\subset \Omega$,

$$\epsilon_0^2 H^{n-2}(\Sigma \cap K) \le \mu(\Sigma \cap K) \le C(K, E) H^{n-2}(\Sigma \cap K).$$

In particular, if $H^{n-2}(\Sigma) = 0$ then $\mu(\Sigma) = 0$ and hence $\nu(\Sigma) = 0$, since $0 \le \nu \le \mu$.

To prove $H^{n-2}(\Sigma) = 0$, it is sufficient to show $H^{n-2}(\Sigma \cap K) = 0$ for any compact $K \subset\subset \Omega$ and hence we may assume that $\Sigma \subset\subset \Omega$ is compact. Since $H^{n-2}(\Sigma) < \infty$, it follows from that (cf. [48]) the 2-capacity of Σ, $\text{cap}_2(\Sigma) = 0$. Therefore, for any $\eta > 0$, there is $\phi_\eta \in C_0^\infty(\Omega, \mathbb{R})$ such that

$$\Sigma \subset \text{int} (\{\phi_\eta = 1\}), \quad \int_\Omega |\nabla \phi_\eta|^2 \le \eta.$$

For any $a \in \Sigma$, there is $0 < \delta_a \le \eta$ such that $\phi_\eta|_{B_{\delta_a}(a)} \ge \frac{1}{2}$. Since Σ is compact and $\Sigma \subset \bigcup_{a \in \Sigma} B_{\frac{\delta_a}{5}}(a)$, there exists $0 < k_\eta < \infty$ such that $\{B_{\frac{\delta_{a_k}}{5}}(a_k)\}_{k=1}^{k_\eta}$ is an open cover of Σ. By Vitali's covering theorem (cf. [55, 48], there is $0 < l_\eta \le k_\eta$ and mutually disjoint balls $\{B_{\frac{\delta_{a_l}}{5}}(a_l)\}_{l=1}^{l_\eta}$ such that

$$\Sigma \subset \bigcup_{1\le l\le l_\eta} B_{\delta_{a_l}}(a_l).$$

By the definition of Σ, there exists a sufficiently large $i_\eta > 0$ such that

$$\left(\frac{\delta_{a_l}}{5}\right)^{2-n} \int_{B_{\frac{\delta_{a_l}}{5}}(a_l)} |\nabla u_i|^2 \ge \frac{\epsilon_0^2}{2}, \ \forall i \ge i_\eta, 1 \le l \le l_\eta.$$

Hence for $i \ge i_\eta$, we have

$$\begin{aligned} H^{n-2}_\eta(\Sigma) &\le \sum_{1\le l\le l_\delta} \delta_{a_l}^{n-2} \le \frac{2\cdot 5^{n-2}}{\epsilon_0^2} \sum_{1\le l\le l_\delta} \int_{B_{\frac{\delta_{a_l}}{5}}(a_l)} |\nabla u_i|^2 \\ &= \frac{2\cdot 5^{n-2}}{\epsilon_0^2} \int_{\cup_{1\le l\le l_\eta} B_{\frac{\delta_{a_l}}{5}}(a_l)} |\nabla u_i|^2. \end{aligned}$$

From (3.75), we have

$$\begin{aligned} \int_{\cup_{1\le\le l\le l_\eta} B_{\frac{\delta_{a_l}}{5}}(a_l)} |\nabla u_i|^2 &\le 4\int_\Omega |\nabla u_i|^2 \phi_\eta^2 \\ &\le A^{-1} \int_\Omega |\nabla \phi_\eta|^2 \le A^{-1}\eta. \end{aligned}$$

Therefore

$$H^{n-2}_\eta(\Sigma) \le \frac{8\cdot 5^{n-2}}{A\epsilon_0^2}\eta.$$

Sending $\eta \to 0$, we obtain $H^{n-2}(\Sigma) = 0$.
Since $\{u_i\}$ converges strongly to u, we conclude that u is stationary harmonic map and also satisfies the stability inequality with the same positive constant A. The proof is complete. □

Recently, Lin-Wang [132] have further improved Theorem 3.4.2 in the dimensions $k = 4, 5, 6, 7$. More precisely, we have

Theorem 3.4.5 *For $k \ge 3$, if $u \in W^{1,2}(\Omega, S^k)$ is a stable-stationary harmonic map, then $dim_H(sing(u)) \le n - \hat{d}(k) - 1$, where*

$$\hat{d}(k) = \begin{cases} 3, & k = 3 \\ 4 & k = 4 \\ 5, & 5 \le k \le 9 \\ 6, & k \ge 6. \end{cases}$$

Note that by direct calculations, we have $\hat{d}(k) \geq d(k) + 1$ for $4 \leq k \leq 7$. The proof of Theorem 3.4.5 is based on a new observation on a sharp improved Kato inequality (3.78). First,we need

Lemma 3.4.6 *For $m \geq 2$ and $k \geq 3$, let $\phi \in H^1(S^m, S^k)$ be a harmonic map such that $\bar{\phi}(x) = \phi(\frac{x}{|x|}) : \mathbb{R}^{m+1} \to S^k$ is a stable harmonic map. Then*

$$\int_{S^m} \left\{ |\nabla \eta|^2 + \frac{(m-1)^2}{4}\eta^2 - \frac{k-2}{k}|\nabla \phi|^2 \eta^2 \right\} \geq 0, \tag{3.76}$$

for all $\eta \in C^\infty(S^m)$.

Proof. Note that $\bar{\phi}$ is homogeneous of degree zero. For any $\eta \in C^\infty(S^m)$, let $\eta_2(x) = \eta_1(|x|)\eta(\frac{x}{|x|})$, where $\eta_1 \in C_0^\infty(\mathbb{R}_+)$ satisfies

$$\begin{aligned} \frac{\int_0^\infty \left(\eta_1'(r)\right)^2 r^m \, dr}{\int_0^\infty \eta_1(r)^2 r^{m-2} \, dr} &= \inf \left\{ \frac{\int_0^\infty \left(\tilde{\eta}'(r)\right)^2 r^m \, dr}{\int_0^\infty \tilde{\eta}(r)^2 r^{m-2} \, dr} \mid \tilde{\eta} \in C_0^\infty((0, +\infty)) \right\} \\ &= \frac{(m-1)^2}{4}. \end{aligned}$$

Substituting η_2 it into (3.74) and direct calculations then imply (3.76). □

Now we recall the Bochner formula for smooth harmonic maps between spheres (cf. §1.5).

Lemma 3.4.7 *For $m, k \geq 2$, if $\phi \in C^\infty(S^m, S^k)$ is a harmonic map, then*

$$\begin{aligned} \Delta \left(\frac{1}{2} |\nabla \phi|^2 \right) &= |\nabla^2 \phi|^2 + (m-1)|\nabla \phi|^2 \\ &- \sum_{1 \leq \alpha, \beta \leq m} \left\{ |\nabla_{e_\alpha} \phi|^2 |\nabla_{e_\beta} \phi|^2 - \langle \nabla_{e_\alpha} \phi, \nabla_{e_\beta} \phi \rangle^2 \right\} \end{aligned} \tag{3.77}$$

where $\{e_\alpha\}_{\alpha=1}^m$ is any local orthonormal frame of S^m.

Next we need a sharp improved Kato's inequality, which was also established by Nakajima [149] independently when $m = k = 3$. For $k = 1$, it is a well-known fact for harmonic functions (see Yau [214]).

Lemma 3.4.8 *Let $\phi \in C^\infty(S^m, S^k)$ be a harmonic map. Then*

$$|\nabla^2 \phi|^2 \geq \frac{m}{m-1} \left| \nabla |\nabla \phi| \right|^2. \tag{3.78}$$

Moreover, the equality holds iff ϕ is totally geodesic.

Proof. By choosing normal coordinates at $x_0 \in S^m$ and $\phi(x_0) \in S^k$, we have

$$\left| \nabla^2 \phi \right|^2 (x_0) = \sum_{1 \leq i \leq k} \sum_{1 \leq \alpha, \beta \leq m} \left(\phi^i_{\alpha\beta}(x_0) \right)^2.$$

On the other hand, since ϕ is a harmonic map, we have

$$\sum_{1\le\alpha\le m} \phi^i_{\alpha\alpha}(x_0) = 0, \ \forall 1 \le i \le k.$$

For any $1 \le i \le k$, let $\{\lambda^i_\alpha\}_{1\le\alpha\le m} \subset \mathbb{R}$ be the eigenvalues of $(\phi^i_{\alpha\beta}(x_0))$ such that $|\lambda^i_1| \le \cdots \le |\lambda^i_m|$. Then we have

$$|\nabla^2\phi|^2(x_0) = \sum_{1\le\le i\le k}\sum_{1\le\alpha\le m} (\lambda^i_\alpha)^2,$$

and

$$\sum_{1\le\alpha\le m} \lambda^i_\alpha = 0, \ \forall 1 \le i \le k. \tag{3.79}$$

On the other hand, by the Cauchy-Schwarz inequality and (3.79), we have

$$\sum_{1\le\alpha\le m-1} \left(\lambda^i_\alpha\right)^2 \ge \frac{1}{m-1}\left(\sum_{1\le\alpha\le m-1}\lambda^i_\alpha\right)^2 = \frac{(\lambda^i_m)^2}{m-1}, \ 1 \le i \le k$$

so that

$$\sum_{1\le\alpha\le m} \left(\lambda^i_\alpha\right)^2 \ge \frac{m}{m-1}(\lambda^i_m)^2.$$

By the Releigh quotient formula, we have for $1 \le i \le k$,

$$|\lambda^i_m|^2 = \sup_{\{0\neq v\in\mathbb{R}^m\}} \frac{\sum_{1\le\alpha\le m}\left(\sum_{1\le\beta\le m}\phi^i_{\alpha\beta}(x_0)v_\beta\right)^2}{|v|^2}.$$

Therefore, for $1 \le i \le k$ we have

$$\left|\nabla\phi^i\right|^2(x_0)\left|\nabla^2\phi^i\right|^2(x_0) \ge \frac{m}{m-1}\sum_{1\le\alpha\le m}\left(\sum_{1\le\beta\le m}\phi^i_{\alpha\beta}(x_0)\phi^i_\beta(x_0)\right)^2.$$

Taking sum of (3.4) over i and applying the Cauchy-Schwarz inequality and the Minkowski inequality, we have at x_0,

$$\begin{aligned}
|\nabla\phi|^2|\nabla^2\phi|^2 &= \left(\sum_{1\le i\le k}|\nabla\phi^i|^2\right)\left(\sum_{1\le i\le k}|\nabla^2\phi^i|^2\right) \ge \left(\sum_{1\le i\le k}|\nabla\phi^i||\nabla^2\phi^i|\right)^2 \\
&\ge \frac{m}{m-1}\left\{\sum_{1\le i\le k}\left(\sum_{1\le\alpha\le m}\left(\sum_{1\le\beta\le m}\phi^i_{\alpha\beta}\phi^i_\beta\right)^2\right)^{\frac12}\right\}^2 \\
&\ge \frac{m}{m-1}\sum_{1\le\alpha\le m}\left(\sum_{1\le i\le k}\sum_{1\le\beta\le m}\phi^i_{\alpha\beta}\phi^i_\beta\right)^2 = \frac{m}{m-1}|\langle\nabla^2\phi,\nabla\phi\rangle|^2.
\end{aligned}$$

Since $|\nabla|\nabla\phi||^2 = \frac{|\langle\nabla^2\phi,\nabla\phi\rangle|^2}{|\nabla\phi|^2}$, this yields (3.78).

Observe that equality in (3.4) holds at $x_0 \in S^m$ iff both the Cauchy-Schwarz inequality and the Minkowski inequality are equalities. This implies

(i)

$$\lambda_1^i = \cdots = \lambda_{m-1}^i = -\frac{\lambda_m^i}{m-1}, \ \forall 1 \le i \le k, \tag{3.80}$$

(ii)

$$\frac{|\nabla^2\phi^1|^2}{|\nabla\phi^1|} = \cdots = \frac{|\nabla^2\phi^k|}{|\nabla\phi^k|}, \tag{3.81}$$

(iii) $\nabla\phi^i(x_0)$ is an eigenfunction of $\nabla^2\phi^i(x_0)$ with the eigenvalue λ_m^i, i.e.,

$$\sum_{1\le\beta\le m} \phi_{\alpha\beta}^i(x_0)\phi_\beta^i(x_0) = \lambda_m^i\phi_\alpha^i(x_0), \ \forall 1 \le i \le k, \ 1 \le \alpha \le m, \tag{3.82}$$

(iv)

$$\sum_{1\le\beta\le m} \phi_{\alpha\beta}^i(x_0)\phi_\beta^i(x_0)(= \lambda_m^i\phi_\alpha^i(x_0))$$

is independent of $1 \le i \le k$ and $1 \le \alpha \le m$.

Thus we have that $\nabla\phi^i(x_0) = \mu^i(1, \cdots, 1)$ for some $\mu^i \neq 0$ for $1 \le i \le k$. Note that (i) and (ii) imply

$$\frac{(\lambda_m^1)^2}{(\mu^1)^2} = \cdots = \frac{(\lambda_m^k)^2}{(\mu^k)^2},$$

and (iii) implies

$$\lambda_m^1\mu^1 = \cdots = \lambda_m^k\mu^k.$$

These two identities imply $|\lambda_1^i| = \cdots = |\lambda_m^i|$. Hence we have $\lambda_\alpha^i = 0$ for all $1 \le \alpha \le m, 1 \le i \le k$, i.e. $\nabla^2\phi(x_0) = 0$. This completes the proof. □

With these lemmas, we can prove the non-existence of certain stable tangent maps, namely tangent maps at a singular point of a stable harmonic map. More precisely,

Lemma 3.4.9 *For $m \ge 2$ and $k \ge 3$, if $\phi \in C^\infty(S^m, S^k)$ is a stable tangent map, then $\frac{(m-1)^2}{4m} \ge \frac{k-2}{k}$.*

Proof. Direct calculations give

$$\Delta\left(\frac{1}{2}|\nabla\phi|^2\right) = |\nabla\phi|\,\Delta\,|\nabla\phi| + |\nabla|\nabla\phi||^2\,.$$

This, combined with (3.78), implies

$$\Delta\,|\nabla\phi| \ge \frac{4}{m-1}\left|\nabla|\nabla\phi|^{\frac{1}{2}}\right|^2 + (m-1)\left(|\nabla\phi| - \frac{1}{m}|\nabla\phi|^3\right). \tag{3.83}$$

Integrating (3.83) over S^m, we obtain

$$\int_{S^m}\left(|\nabla|\nabla\phi|^{\frac{1}{2}}|^2 + \frac{(m-1)^2}{4}|\nabla\phi|\right) \le \frac{(m-1)^2}{4m}\int_{S^m}|\nabla\phi|^3. \tag{3.84}$$

On the other hand, (3.76) implies

$$\int_{S^m} \left(|\nabla|\nabla\phi|^{\frac{1}{2}}|^2 + \frac{(m-1)^2}{4}|\nabla\phi| \right) \geq \frac{k-2}{k} \int_{S^m} |\nabla\phi|^3. \tag{3.85}$$

Therefore combining (3.84) with (3.85) yields

$$\left(\frac{k-2}{k} - \frac{(m-1)^2}{4m} \right) \int_{S^m} |\nabla\phi|^3 \leq 0.$$

Since ϕ is nonconstant, this implies $\frac{k-2}{k} \leq \frac{(m-1)^2}{4m}$. Hence the proof is complete. □

Proof of Theorems 3.4.2 and 3.4.5:
It follows from Lemma 3.4.4 that Federer's dimension reduction argument is applicable. This combines with Lemma 3.4.9 immediately implies the dimension estimate of the singular set as stated in these theorems. □

Note that when $m = k = 3$, then Lemma 3.4.9 and Lemma 3.4.8 imply that if a harmonic map $\phi \in C^\infty(S^3, S^3)$ is such that $\phi(\frac{x}{|x|}) : \mathbb{R}^4 \to S^3$ is stable, then ϕ must be conformal and totally geodesic and hence must be an isometry. In particular, we have the following classification theorem (see also [149]).

Theorem 3.4.10 *Suppose that $\phi \in C^\infty(S^3, S^3)$ is a nontrivial harmonic map such that its homogeneous of degree zero extension $\bar\phi(x) = \phi(\frac{x}{|x|}) : \mathbb{R}^4 \to S^3$ is a stable harmonic map. Then*

$$\bar\phi(x) = Q(\frac{x}{|x|}) \tag{3.86}$$

for some orthogonal rotation $Q \in O(3)$.

As a direct consequence of Theorem 3.4 is the following useful fact on stable-stationary harmonic maps from $\mathbb{R}^4$ to S^3.

Proposition 3.4.11 *For a bounded domain $\Omega \subset \mathbb{R}^4$, if $u \in H^1(\Omega, S^3)$ is a stable-stationary harmonic map, then sing(u) is discrete, and for any $x_0 \in$ sing(u),*

$$\Theta^2(u, x_0) = \lim_{r \to 0} r^{-2} \int_{B_r(x_0)} |\nabla u|^2 = \frac{3}{2}|S^3|. \tag{3.87}$$

If there exists $r_0 > 0$ such that $\int_{B_{r_0}(x_0)} |\nabla u|^2 = \frac{3}{2}|S^3|r_0^2$, then there exists $Q \in O(3)$ such that $u(x) = Q\left(\frac{x-x_0}{|x-x_0|}\right)$ for $x \in B_{r_0}(x_0)$.

Proof. The discreteness of sing(u) follows from Theorem 3.4.2. Moreover, it follows from (3.42) and Lemma 3.4.4 that for any $x_0 \in$ sing(u) and $r_i \to 0$, there exists a nontrivial smooth harmonic map $\phi \in C^\infty(S^3, S^3)$ such that, after taking possible subsequences,

$$\lim_{i\to\infty} \left\| u\left(x_0 + r_i x\right) - \phi\left(\frac{x}{|x|}\right) \right\|_{H^1(B_R)} = 0, \ \forall R > 0.$$

Hence $\phi\left(\frac{x}{|x|}\right) : \mathbb{R}^4 \to S^3$ is a stable harmonic map. Therefore implies $\phi = Q$ for some $Q \in O(3)$ and

$$\Theta^2(u, x_0) = \int_{B_1} \left|\nabla\left(Q(\frac{x}{|x|})\right)\right|^2 = \int_{B_1} \left|\nabla\left(\frac{x}{|x|}\right)\right|^2 = 3(\int_0^1 r\,dr)|S^3| = \frac{3}{2}|S^3|.$$

This implies (3.87). The second part follows from (3.42) and the first part easily. □

With Lemma 3.4.11, we can extend the corresponding theorems on the interior energy bound on minimizing harmonic maps from B^3 to S^2 by Hardt-Lin [84] and Almgren-Lieb [6] to stable-stationary harmonic maps from B^4 to S^3.

Theorem 3.4.12 *Assume that $u \in H^1(B^4, S^3)$ is a stable-stationary harmonic map. Then for any $0 < \epsilon < 1$, there exist positive constants K_ϵ, L_ϵ such that*
(i)

$$r^{2-n} \int_{B_r(a)} |\nabla u|^2 \le K_\epsilon, \ \forall B_r(a) \subset B_{1-\epsilon}, \tag{3.88}$$

and (ii) the number of singularities of u in $B_{1-\epsilon}$ is bounded by L_ϵ.

Proof. Since $u \in H^1(B^4, \mathbf{S}^3)$ is a stable harmonic map so that (3.74) implies

$$\int_{B^4} |\nabla u|^2 \eta^2 \le \frac{1}{3} \int_{B^4} |\nabla \eta|^2, \forall \eta \in C_0^1(B^4).$$

This implies, for any $0 < \epsilon < 1$,

$$\int_{B_{1-\epsilon}} |\nabla u|^2 \le C\epsilon^{-1}.$$

Therefore (i) follows. To show (ii), we may assume for simplicity $\epsilon = \frac{1}{2}$ and prove that the number of singular points of u inside $B_{\frac{1}{2}}$ is uniformly bounded. This follows if we can prove that there exists a universal constant $\delta_0 > 0$ such that the distance between any two singular points of u inside $B_{\frac{1}{2}}$ is at least δ_0. Suppose that there were false. Then we may assume that there exists a sequence of stable-stationary harmonic maps $u_i \in H^1(B^4, S^3)$ and a sequence of points $x_i \in B_{\frac{1}{2}}$ with $|x_i| \to 0$ such that $\{0, x_i\} \subset \mathrm{sing}(u_i)$. Now we claim that there exists another universal constant $\delta_1 > 0$ such that

$$\int_{B_{2|x_i|}} |\nabla u_i|^2 \ge \left(\frac{3}{2}|\mathbf{S}^3| + \delta_1\right)(2|x_i|)^2.$$

For otherwise, the rescaled maps $v_i(x) = u_i(|x_i|x) : B_2 \to S^3$ satisfy

$$\lim_{i\to\infty} 2^{-2} \int_{B_2} |\nabla v_i|^2 = \frac{3}{2}|S^3|$$

so that there exist a stable-stationary harmonic map $v_\infty \in H^1(B_4, S^3)$ and a point $x_\infty \in S^3$ such that $v_i \to v_\infty$ in $H^1(B_2)$, $\{0, x_\infty\} \in \mathrm{sing}(v_\infty)$, and

$$2^{-2} \int_{B_2} |\nabla v_\infty|^2 = \frac{3}{2}|S^3|.$$

This, combined with Proposition 3.4.11, implies $v_\infty(x) = Q(\frac{x}{|x|})$ in B_2 for some $Q \in O(3)$ and $\text{sing}(v_\infty) = \{0\}$. We get the desired contradiction and complete the proof. □

Remark 3.4.13 In general, it is an interesting question to find the existence of stable-stationary harmonic maps even in lower dimensions (e.g. from B^3 to S^2). We speculate that the method of relaxed energy for harmonic maps [14, 69, 70] might be a possible approach to attack this issue.

Chapter 4

Blow up analysis of stationary harmonic maps

In this chapter, we will study general properties of sequences of weakly convergent stationary harmonic maps between two Riemannian manifolds (M, g) and (N, h). More precisely, we will analyze the defect measures and energy concentration sets associated with such weakly convergent sequences of stationary harmonic maps. This includes (i) rectifiability of the energy concentration set, (ii) necessary and sufficient conditions under which the set of stationary harmonic maps is compact in $H^1(M, N)$, and (iii) both necessary and sufficient conditions for uniform interior gradient estimates for stationary harmonic maps in terms of their total energy, namely, under what conditions on the target manifold N that any stationary harmonic map $u : M \to N$ is smooth and satisfies the estimate

$$\|u\|_{C^k(M)} \leq C\,(M, N, k, E) \quad \text{for } k \geq 1, \text{ where } E = \int_M |\nabla u|^2. \tag{4.1}$$

The results presented here are from Lin [123].

4.1 Preliminary analysis

The first step to analyze the singular set of stationary harmonic maps is to understand the behavior of a sequence of weakly convergent stationary harmonic maps. We start with an example that illustrates the strong convergence may fail.

Example 4.1.1 Let $v : S^2 \to S^2$ be any nontrivial conformal map and $\Phi : \overline{\mathbb{R}^2} \to S^2$ be the inverse of the stereographic projection from S^2 to $\overline{\mathbb{R}^2}$. Then $u = v \circ \Phi : \mathbb{R}^2 \to S^2$ is a smooth harmonic map. For $\lambda > 0$, let $u_\lambda(x) = u(\lambda x), x \in \mathbb{R}^2$. Then $u_\lambda \to$ constant $= u(\infty)$ weakly *but not* strongly in $H^1_{\text{loc}}(\mathbb{R}^2, S^2)$ as $\lambda \to 0$, since $E(u_\lambda) = 8\pi|\deg(v)|(> 0)$ for any $\lambda > 0$. In fact, $|\nabla u_\lambda|^2\, dx \to 8\pi|\deg(v)|\delta_0$ as convergence of Radon measures, as $\lambda \to 0$. Now if we view u, u_λ as smooth harmonic maps from $\mathbb{R}^n$ $(n \geq 3)$ to S^2 that are independent of variables $x_3, \cdots, x_n$, then

$$u_\lambda \to \text{constant weakly in } H^1(\mathbb{R}^n),$$

$$|\nabla u_\lambda|^2\,dx \to 8\pi|\deg(v)|H^{n-2}\lfloor\left(\{(0,0)\}\times\mathbb{R}^{n-2}\right)$$

as convergence of Radon measures.

For simplicity, we assume throughout this chapter that M is a bounded domain $\Omega\subset\mathbb{R}^n$. Let $u\in H^1(\Omega,N)$ be a stationary harmonic map. Then we have the stationarity identity (see also (3.41)):

$$\int_\Omega\sum_{i,j=1}^n\left(\delta_{ij}|\nabla u|^2-2\nabla_i u\nabla_j u\right)\nabla_i\xi^j=0,\ \ \xi\in C_0^\infty(\Omega,\mathbb{R}^n), \tag{4.2}$$

By choosing suitable ξ as §3.3, we obtain

$$(n-2)\int_{B_\rho(x)}|\nabla u|^2=\rho\int_{\partial B_\rho(x)}\left(|\nabla u|^2-2|u_{R_x}|^2\right),\ \text{a.e. } 0<\rho<\ \text{dist}(x,\partial\Omega), \tag{4.3}$$

where

$$u_{R_x}(y)=\frac{(y-x)}{|y-x|}\cdot\nabla u.$$

Hence by intergration we have

$$\rho^{2-n}\int_{B_\rho(x)}|\nabla u|^2-\sigma^{2-n}\int_{B_\sigma(x)}|\nabla u|^2=2\int_{B_\rho(x)\setminus B_\sigma(x)}\frac{|R_x u_{R_x}|^2}{R_x^n} \tag{4.4}$$

for any $0<\sigma\le\rho<\text{dist}(x,\partial\Omega)$, where $R_x(y)=|y-x|$. In particular,

$$\rho^{2-n}\int_{B_\rho(x)}|\nabla u|^2$$

is monotonically non-decreasing with respect to ρ so that

$$\Theta^{n-2}(u,x)=\lim_{\rho\downarrow 0}\rho^{2-n}\int_{B_\rho(x)}|\nabla u|^2 \tag{4.5}$$

exists and is upper semicontinuous for all $x\in\Omega$. Letting $\sigma\to 0$, (4.4) yields

$$\rho^{2-n}\int_{B_\rho(x)}|\nabla u|^2-\Theta^{n-2}(u,x)=2\int_{B_\rho(x)}\frac{|R_x u_{R_x}|^2}{R_x^n}. \tag{4.6}$$

This, combined with (4.3), implies

$$2\int_{B_\rho(x)}\frac{|R_x u_{R_x}|^2}{R_x^n}\le\frac{1}{n-2}\rho^{3-n}\int_{\partial B_\rho(x)}|\nabla u|^2\,dH^{n-1}-\Theta^{n-2}(u,x) \tag{4.7}$$

for any $B_\rho(x)\subset\Omega$.

Denote by $\text{sing}(u)$ the singular set of u. Then the ϵ_0-regularity lemma 3.3.14 implies

$$x\in\text{sing}(u)\iff\Theta^{n-2}(u,x)\ge\epsilon_0^2\iff\Theta^{n-2}(u,x)>0. \tag{4.8}$$

For any $\Lambda>0$, let H_Λ be the set of stationary harmonic maps u from Ω to N such that $E(u,\Omega)\le\Lambda$.

Based on (4.4) and (4.8), we can modify the argument for smooth harmonic maps by Schoen [166] to show

Proposition 4.1.2 *Any map u in the closure of H_Λ with respect to weak convergence in $H^1(\Omega, N)$ is weakly harmonic and smooth outside a closed set $\Sigma \subset \Omega$ with locally finite $(n-2)$-dimensional Hausdorff measure.*

Proof. Let $u_i \in H_\Lambda$ be such that $u_i \to u$ weakly in $H^1(\Omega, N)$. Consider a sequence of Radon measures $\mu_i = |\nabla u_i|^2\, dx, i = 1, 2, \cdots$. Then we may assume that there is a nonnegative Radon measure μ in Ω such that $\mu_i \to \mu$ as convergence of Radon measures. By Fatou's lemma, we can write

$$\mu = |\nabla u|^2\, dx + \nu$$

for a nonnegative Radon measure ν, called *defect* measure.
Since each u_i satisfies (4.4), $\rho^{2-n}\mu(B_\rho(x))$ is monotonically non-decreasing in ρ for any $x \in \Omega$. Hence

$$\Theta^{n-2}(\mu, x) \equiv \lim_{\rho \downarrow 0} \rho^{2-n}\mu(B_\rho(x))$$

exists and is upper semicontinuous for all $x \in \Omega$. Define

$$\begin{aligned} \Sigma &= \bigcap_{r>0} \left\{ x \in \Omega \mid \liminf_{i\to\infty} r^{2-n} \int_{B_r(x)} |\nabla u_i|^2 \geq \epsilon_0^2 \right\} \\ &= \left\{ x \in \Omega : \Theta^{n-2}(\mu, x) \geq \epsilon_0^2 \right\}. \end{aligned} \tag{4.9}$$

The closeness of Σ follows from the upper semiconutinuity of $\Theta^{n-2}(\mu, \cdot)$.
Suppose $x_0 \in \Omega \setminus \Sigma$, then there is $r_0 > 0$ such that

$$\liminf_{i\to\infty} r_0^{2-n} \int_{B_{r_0}(x_0)} |\nabla u_i|^2 < \epsilon_0^2.$$

Hence Lemma 3.3.14 implies that there is a subsequence $\{i'\} \subset \{i\}$ such that

$$\sup_{B_{\frac{r_0}{2}}(x_0)} |\nabla u_{i'}| \leq \frac{C\epsilon_0}{r_0}$$

for i' sufficiently large. We may assume that $u_{i'} \to u$ in $C^2\left(B_{\frac{r_0}{4}}(x_0), N\right)$. Hence, after passing to a subsequence, $u_i \to u$ in $C^2_{\text{loc}} \cap H^1_{\text{loc}}(\Omega \setminus \Sigma, N)$.
Note that for any compact subset $K \subset\subset \Omega$, by (4.4) we have

$$\epsilon_0^2 \leq \Theta^{n-2}(\mu, x) \leq \delta_K^{2-n}\mu(\Omega), \ \forall x \in \Sigma \cap K, \tag{4.10}$$

where $\delta_K = \text{dist}(K, \partial\Omega)$. Therefore it is readily seen that

$$H^{n-2}(\Sigma \cap K) \leq \frac{C(n, K)}{\epsilon_0^2}\mu(\Omega) \leq C(n, K, \epsilon_0, \Lambda). \tag{4.11}$$

A simple capacity argument (see §7 below) yields that u is a weakly harmonic map in Ω. □

Since $u \in H^1(\Omega, N)$, it is well-known (see [56]) that for H^{n-2} a.e. $x \in \Omega$,

$$\Theta^{n-2}(u, x) \equiv \lim_{r\to 0} r^{2-n} \int_{B_r(x)} |\nabla u|^2 = 0. \tag{4.12}$$

As a consequenc of (4.10) and (4.12), we have

Lemma 4.1.3 *On a closed ball* $B_{1+\delta_0} \subset \Omega$,
(1) $\Sigma = supp(\nu) \cup sing(u)$,
(2) $\nu(x) = \Theta(x)H^{n-2}\lfloor\Sigma$ *for* $x \in B_1$, *where* $\epsilon_0^2 \leq \Theta(x) \leq C(\delta_0, n)\Lambda$ *for* H^{n-2} *a.e.* $x \in \Sigma$.

Proof. (1) is trivial. It follows from (4.10) that $\mu\lfloor\Sigma$ is absolutely continuous with respect to $H^{n-2}\lfloor\Sigma$. Hence, by Radon-Nikodym theorem, we have that there exist a measurable function Θ such that

$$\mu\lfloor\Sigma = \Theta(x)H^{n-2}\lfloor\Sigma.$$

This, combined with (4.12), implies

$$\nu(x) = \Theta(x)H^{n-2}\lfloor\Sigma$$

for H^{n-2} a.e. $x \in \Sigma$. By the definition of Σ, one has $\Theta(x) \geq \epsilon_0^2$ for H^{n-2} a.e. $x \in \Sigma$. The upper bound of Θ follows from both (4.11) and that for H^{n-2} a.e. $x \in \Sigma$ (see for example [189])

$$2^{2-n} \leq \limsup_{r\downarrow 0} \frac{H^{n-2}(\Sigma \cap B_r(x))}{r^{n-2}} \leq 1.$$

□

To explore properties of Σ and μ, assume $B_1 \subset B_{1+\delta_0} = \Omega$. Denote by $\mathcal{M}$ the set of all Radon measures μ on B_1 such that μ is a weak limit of Radon measures $\mu_i = |\nabla u_i|^2\,dx$ on Ω, where $\{u_i\} \subset H_\Lambda$ for some $\Lambda > 0$. Also denote by $\mathcal{F}$ the set of all compact subset E of B_1 such that $E \subset \Sigma$ for some Σ defined by (4.9) associated with $\{u_i\} \subset H_\Lambda$ for some $\Lambda > 0$.

From proposition 4.1.2, we know that for $\mu \in \mathcal{M}$, there are $\Sigma \in \mathcal{F}$, a weakly harmonic map $u \in H^1(B_1, N) \cap C^\infty(B_1 \setminus \Sigma, N)$ and a nonnegative Radon measure ν on B_1 such that $\mu = |\nabla u|^2\,dx + \nu$ in B_1.

For $E \in \mathcal{F}$ and $\mu \in \mathcal{M}$, if $y \in B_1$ and $0 < \lambda < 1 - |y|$ then we define

$$E_{y,\lambda} = \lambda^{-1}(E - y) \cap B_1,$$

$$\mu_{y,\lambda}(A) = \lambda^{2-n}\mu(y + \lambda A),\ \forall A \subset B_1.$$

Then one has

Lemma 4.1.4 *(i) For* $y \in B_1$ *and* $0 < \lambda < 1 - |y|$, *if* $E \in \mathcal{F}$ *and* $\mu \in \mathcal{M}$, *then* $E_{y,\lambda} \in \mathcal{F}$ *and* $\mu_{y,\lambda} \in \mathcal{M}$.
(ii) For $\mu \in \mathcal{M}$, $y \in B_1$ *and* $\lambda_k \downarrow 0$, *there is* $\eta \in \mathcal{M}$ *such that, after taking subsequences,* $\mu_{y,\lambda_k} \to \eta$. *Moreover,* $\eta_{0,\lambda} = \eta$ *for any* $\lambda > 0$.
(iii) For $E \in \mathcal{F}$, $y \in B_1$ *and* $\lambda_k \downarrow 0$, *there exists* $F \in \mathcal{F}$ *such that, after taking subsequence,* $E_{y,\lambda_k} \to F$ *in the Hausdorff metric. Moreover,* $F \subset \Sigma_*$ *for some* Σ_* *defined as (4.9) and* $(\Sigma_*)_{0,\lambda} = \Sigma_*$ *for any* $\lambda > 0$.
(iv) $\mathcal{M}$ *is closed with respect to the weak convergence of measures.*
(v) Define a map $\pi : \mathcal{M} \to \mathcal{F}$ *as follows:*

$$\pi(\mu) = \begin{cases} \Sigma & \text{if } \mu = |\nabla u|^2\,dx + \nu \text{ and } 0 \neq \nu = \Theta(x)H^{n-2}\lfloor\Sigma \\ sing(u) & \text{if } \nu = 0 \text{ and } \mu = |\nabla u|^2\,dx. \end{cases}$$

Then
(a) For any $|y| < 1$ *and* $0 < \lambda < 1 - |y|$*, if* $\mu \in \mathcal{M}$ *then*

$$\pi(\mu_{y,\lambda}) = \lambda^{-1}(\pi(\mu) - y) \ \text{ on } B_1.$$

(b) If $\mu_k, \mu \in \mathcal{M}$ *satisfies* $\mu_k \to \mu$*, then for any* $\epsilon > 0$ *there is* $k(\epsilon) > 0$ *such that*

$$\pi(\mu_k) \cap B_1 \subset \{x \in B_{1+\delta_0} : dist\,(x, \pi(\mu)) < \epsilon\} \ \text{ for } k \geq k(\epsilon). \tag{4.13}$$

Proof. (i) is trivial. In fact, let $\{u_i\} \subset H_\Lambda$ be such that

$$\mu_i = |\nabla u|^2\, dx \to \mu.$$

Then $u_{i,y,\lambda}(x) = u_i(y + \lambda x)$ is defined for $x \in B_{1+\delta_0}$ and

$$\int_{B_{1+\delta_0}} |\nabla u_{i,y,\lambda}|^2 = \lambda^{2-n} \int_{B_{\lambda(1+\delta_0)}(y)} |\nabla u_i|^2 \leq \left(\frac{1+\delta_0}{\delta_0}\right)^{n-2} \Lambda.$$

This implies $u_{i,y,\lambda} \in H_{\hat{\Lambda}}$, with $\hat{\Lambda} = \left(\frac{1+\delta_0}{\delta_0}\right)^{n-2} \Lambda$. Since

$$|\nabla u_{i,y,\lambda}|^2\, dx \to \mu_{y,\lambda}$$

as convergence of Radon measures, we have $\mu_{y,\lambda} \in \mathcal{M}$. The statement on $E \in \mathcal{F}$ can be proved similarly.
To prove (ii), observe that for any $\lambda_k \downarrow 0$, one has

$$\lim_{k\to\infty} \mu_{y,\lambda_k}(B_R) = R^{n-2}\Theta^{n-2}(\mu, y), \ \forall R > 0. \tag{4.14}$$

Therefore we may assume that there is a nonnegative Radon measure η in $\mathbb{R}^n$ such that $\mu_{y,\lambda_k} \to \eta$ as convergence of Radon measures. Moreover, by a diagonal process, there exists a subsequence $\{i_k\} \subset \{i\}$ such that

$$|\nabla u_{i_k,y,\lambda_k}|^2\, dx \to \eta.$$

Hence $\eta \lfloor B_1 \in \mathcal{M}$.
By (4.14), we have

$$R^{2-n}\eta\,(B_R(0)) = \Theta^{n-2}\,(\mu, y) \ \text{ for all } R > 0. \tag{4.15}$$

We can follow [123] (page 796-800) to show that η is a cone measure, i.e. $(\eta)_{0,\lambda} = \eta$ for $\lambda > 0$. Here we present a short proof (see also Moser [147]). Denote $v^k = u_{i_k,y,\lambda_k}$ and set $\mu^k_{\alpha\beta} = \left\langle \nabla_\alpha v^k, \nabla_\beta v^k \right\rangle dx$ for $1 \leq \alpha, \beta \leq n$. Since $|\mu^k_{\alpha\beta}|(\Omega)$ is uniformly bounded, we may assume that there is (signed) Radon measure $\mu_{\alpha\beta}$ on Ω such that $\mu^k_{\alpha\beta} \to \mu_{\alpha\beta}$ as $k \to \infty$ as convergence of Radon measures. Moreover, since $\sum_{\alpha=1}^n \mu^k_{\alpha\alpha} = |\nabla u_{i_k,y,\lambda_k}|^2\, dx$, we have

$$\eta = \sum_{\alpha=1}^n \mu_{\alpha\alpha} \text{ in } \Omega.$$

Since v^k satisfies the monotonicity formula: for any $0 < r \le R < +\infty$ and $x_0 \in \mathbb{R}^n$,

$$\begin{aligned} & R^{2-n}\int_{B_R(x_0)} |\nabla v^k|^2 - r^{2-n}\int_{B_r(x_0)} |\nabla v^k|^2 \\ = \; & 2\sum_{\alpha,\beta=1}^n \int_{B_R(x_0)\setminus B_r(x_0)} |x-x_0|^{-n}(x-x_0)_\alpha(x-x_0)_\beta \langle \nabla_\alpha v^k, \nabla_\beta v^k\rangle \, dx \ge 0, \end{aligned}$$

we have, by taking $k \to \infty$,

$$\begin{aligned} & R^{2-n}\eta\left(B_R(x_0)\right) - r^{2-n}\eta\left(B_r(x_0)\right) \\ & = 2\sum_{\alpha,\beta=1}^n \int_{B_R(x_0)\setminus B_r(x_0)} |x-x_0|^{-n}(x-x_0)_\alpha(x-x_0)_\beta \, d\mu_{\alpha\beta} \ge 0. \quad (4.16) \end{aligned}$$

Taking $x_0 = 0$ and using (4.15), (4.16) implies

$$\sum_{\alpha\beta=1}^n \mu_{\alpha\beta}\lfloor x_\alpha x_\beta = 0.$$

Note that for any $\phi \in C_0^\infty(\mathbb{R}^n, \mathbb{R}^n)$, since

$$\left(\sum_{\alpha,\beta=1}^n \int_{\mathbb{R}^n} x_\alpha \phi_\beta \, d\mu^k_{\alpha\beta}\right)^2 \le \left(\sum_{\alpha,\beta=1}^n \int_{\mathbb{R}^n} \phi_\alpha\phi_\beta \, d\mu^k_{\alpha\beta}\right)\cdot\left(\sum_{\alpha,\beta=1}^n \int_{\text{supp}(\phi)} x_\alpha x_\beta \, d\mu^k_{\alpha\beta}\right)$$

we have, by taking $k \to \infty$,

$$\sum_{\alpha,\beta=1}^n \int_{\mathbb{R}^n} x_\alpha \phi_\beta \, d\mu_{\alpha\beta} = 0. \tag{4.17}$$

For any $\psi \in C_0^1(B_1)$ and $\lambda > 0$, define $\psi_\lambda(x) = \psi(\frac{x}{\lambda})$. Since v^k satisfies the identity (4.2), we have

$$\sum_{\alpha,\beta=1}^n \int_{\mathbb{R}^n} \left(\delta_{\alpha\beta}|\nabla v^k|^2 - 2\langle \nabla_\alpha v^k, \nabla_\beta v^k\rangle\right) \nabla_\alpha\left(\psi_\lambda x_\beta\right) = 0.$$

Taking $k \to \infty$ and applying (4.17) with ϕ_β replaced by $\nabla_\beta(\psi_\lambda)$, we obtain

$$\int_{\mathbb{R}^n} \left((n-2)\psi_\lambda + \lambda^{-1}x\cdot\nabla\psi(\frac{x}{\lambda})\right) d\eta = 2\sum_{\alpha,\beta=1}^n \int_{\mathbb{R}^n} x_\alpha \nabla_\beta \psi_\lambda \, d\mu_{\alpha\beta} = 0.$$

Since

$$\begin{aligned} \frac{d}{d\lambda}\left(\eta_{0,\lambda}(\psi)\right) &= \frac{d}{d\lambda}\left(\lambda^{2-n}\int_{\mathbb{R}^n} \psi_\lambda \, d\eta\right) \\ &= -\lambda^{1-n}\int_{\mathbb{R}^n} \left((n-2)\psi_\lambda + \lambda^{-1}x\cdot\nabla\psi(\frac{x}{\lambda})\right) d\eta, \end{aligned}$$

we conclude that $\frac{d}{d\lambda}(\eta_{0,\lambda}(\psi)) = 0$ and hence $\eta_{0,\lambda} = \eta$ for any $\lambda > 0$.
(iii) and (iv) follow from (ii) easily, and (a) of (v) is also obvious. We sketch the proof for (b) of (v). First, by Blaschike's selection principle [55], we may assume that after taking possible subsequences

$$\pi(\mu_k) \to F$$

for some closed subset $F \subset B_1$ in the Hausdorff metric. Therefore for any $\epsilon > 0$ there is $k(\epsilon) > 0$ such that for $k \geq k(\epsilon)$,

$$\pi(\mu_k) \cap B_1 \subset \{x \in B_{1+\delta_0} \mid \operatorname{dist}(x, F) < \epsilon\}.$$

It suffices to show $F \subset \pi(\mu)$. For any $x \in F$, there is $x_k \in \pi(\mu_k)$ such that $x_k \to x$. Note that (4.16) implies that for any $y \in B_{1+\delta_0}$, $\Theta_r^{n-2}(\mu_k, y) \equiv r^{2-n}\mu_k(B_r(y))$ is monotonically nondecreasing. Hence $\Theta^{n-2}(\mu_k, y) = \lim_{r\downarrow 0}\Theta_r^{n-2}(\mu_k, y)$ exists for any $y \in B_{1+\delta_0}$ and is upper semicontinuous with respect to both μ_k and y variables. In particular,

$$\Theta^{n-2}(\mu, x) \geq \limsup_{k\to\infty} \Theta^{n-2}(\mu_k, x_k).$$

On the other hand, $x_k \in \pi(\mu_k)$ if and only if $\Theta^{n-2}(\mu_k, x_k) \geq \epsilon_0^2$. Therefore $\Theta^{n-2}(\mu, x) \geq \epsilon_0^2$ and $x \in \pi(\mu)$. The proof is complete. □

For any $\mu \in \mathcal{M}$ and $x_0 \in \pi(\mu)$, we say $\eta \in \mathcal{M}$ is a *tangent* measure of μ at x_0 if there exists $\lambda_k \downarrow 0$ such that

$$\mu_{x_0,\lambda_k} \to \eta$$

as convergence of Radon measures. Denote by $T_{x_0}\mu$ the set of all tangent measures of μ at x_0. By Lemma 4.1.4 (ii), we have that any $\eta \in T_{x_0}\mu$ is a cone measure. It is also not hard to see (4.16) implies that for any $\eta \in T_{x_0}\mu$,

$$\Theta^{n-2}(\eta, 0) = \Theta^{n-2}(\mu, x_0) = \max\left\{\Theta^{n-2}(\eta, y) : y \in \mathbb{R}^n\right\}. \tag{4.18}$$

Moreover, if $0 \neq y \in \mathbb{R}^n$ is such that

$$\Theta^{n-2}(\eta, y) = \Theta^{n-2}(\eta, 0),$$

then η satisfies $\eta_{y,\lambda} = \eta$ for any $\lambda > 0$. In particular, as in §2.3, we have that η is invariant in y-direction, i.e. $\eta_{y,1} = \eta$. Hence

$$L_\eta = \left\{y \in \mathbb{R}^n : \Theta^{n-2}(\eta, y) = \Theta^{n-2}(\eta, 0)\right\}$$

is a linear subspace of $\mathbb{R}^n$ and $\dim(L_\eta) \leq n-2$ (since $\Theta^{n-2}(\eta, 0)$ is finite). Moreover, if $\dim(L_\eta) = n-2$ then we may identify $L_\eta = \mathbb{R}^{n-2} \times \{(0,0)\} \subset \mathbb{R}^n$ so that

$$\eta = \Theta^{n-2}(\mu, x_0) H^{n-2} \lfloor \left(\mathbb{R}^{n-2} \times \{(0,0)\}\right).$$

As in §2.3, we can define a stratification of $\Sigma = \pi(\mu)$ for $\mu \in \mathcal{M}$ as follows. For $0 \leq j \leq n-2$,

$$\Sigma_j = \{x \in \Sigma : \dim(L_\eta) \leq j \text{ for all } \eta \in T_{x_0}\mu\}.$$

Then we have

$$\Sigma_0 \subset \Sigma_1 \subset \cdots \subset \Sigma_{n-2} = \Sigma. \tag{4.19}$$

Moreover, the same proof of Lemma 2.3.3 gives

Proposition 4.1.5 *Let* $\mu \in \mathcal{M}$ *and* $\Sigma = \pi(\mu)$. *Then* $\Sigma = \cup_{j=0}^{n-2} \Sigma_j$, *and* $dim_H(\Sigma_j) \leq j$ *for* $j = 0, \cdots, n-2$.

4.2 Rectifiability of defect measures

From the previous section, we know that any $\mu \in \mathcal{M}$ can be written as

$$\mu = |\nabla u|^2 \, dx + \nu$$

where $u \in H^1(\Omega, N)$ is a weakly harmonic map that is smooth away from the concentration set Σ of locally finite H^{n-2}-measure, and ν is a defect measure such that

$$\nu(x) = \Theta(x) H^{n-2} \lfloor \Sigma, \ \epsilon_0^2 \leq \Theta(x) \leq C(\mu) < +\infty \ \text{ for } H^{n-2} \text{ a.e. } x \in \Sigma.$$

The purpose of this section is to show that Σ is a $(n-2)$-dimensional rectifiable set and ν is a H^{n-2}-rectifiable measure. First we recall (see [55, 189])

Definition 4.2.1 For $l = 0, 1, \cdots, n$, a set $E \subset \mathbb{R}^n$ is called l-rectifiable if $E \subset \cup_{j=0}^{\infty} N_j$, where $H^l(N_0) = 0$ and each N_j, $j \geq 1$, is an l-dimensional embedded C^1 submanifold of $\mathbb{R}^n$.

Definition 4.2.2 For $l = 0, 1, \cdots, n$, if $E \subset \mathbb{R}^n$ is H^l-measurable and θ is a positive locally H^l-integrable function on E, then we say that a l-dimensional subspace $P \subset \mathbb{R}^n$ is the approximate tangent space of E at $x \in \mathbb{R}^n$ with respect to θ, if

$$\lim_{\lambda \downarrow 0} \int_{E_{x,\lambda}} f(y) \theta(x + \lambda y) \, dH^l(y) = \theta(x) \int_P f \, dH^l \ \forall f \in C_0(\mathbb{R}^n),$$

or equivalently

$$\lim_{\lambda \downarrow 0} \lambda^{-l} \int_E f(\lambda^{-1}(z - x)) \theta(z) \, dH^l(z) = \theta(x) \int_P f \, dH^l \ \forall f \in C_0(\mathbb{R}^n).$$

The following theorem characterizes the rectifiability of a set in terms of approximate tangent spaces.

Theorem 4.2.3 *For* $l = 0, 1, \cdots, n$, *suppose* $E \subset \mathbb{R}^n$ *is* H^l*-measurable. Then* E *is* l*-rectifiable if and only if there is a positive locally* H^l*-integrable fucntion* θ *on* E *with respect to which the approximate tangent space of* E *exists for* H^l *a.e.* $x \in E$.

We also recall

Definition 4.2.4 For $l = 0, 1, \cdots, n$, a nonnegative Radon measure μ on $\mathbb{R}^n$ is called to be l-rectifiable if there is a l-rectifiable set $\Sigma \subset \mathbb{R}^n$ and a positive locally H^l-integrable function θ on Σ such that $\mu = \theta H^l \lfloor \Sigma$.

The main theorem of this section, due to Lin [123], is

Theorem 4.2.5 *For any $\mu = |\nabla u|^2\, dx + \nu \in \mathcal{M}$, the concentration set $\pi(\mu)$ is an $(n-2)$-rectifiable set and the defect measure ν is an $(n-2)$-rectifiable set.*

The crucial step to prove this theorem is the following geometric Lemma.

Lemma 4.2.6 *Let $x \in \Sigma$ be such that $\Theta^{n-2}(\mu, y)$ is H^{n-2}-approximate continuous at x, for $y \in \Sigma$. Then there exist $s \in (0, \frac{1}{2})$ depending only on n and $r_x > 0$ such that for any $0 < r \le r_x$ there are $(n-2)$ points $\{x_1, \cdots, x_{n-2}\} \subset B_r(x) \cap \Sigma$ and $\epsilon(r) > 0$ with $\lim_{r\downarrow 0} \epsilon(r) = 0$ such that*
(i) $\Theta^{n-2}(\mu, x_i) \ge \Theta^{n-2}(\mu, x) - \epsilon(r)$ for $j = 1, 2, \cdots, n-2$,
(ii) $|x_1| \ge sr$ and for any $k \in \{2, \cdots, n-2\}$, $dist(x_k, x + V_{k-1}) \ge sr$, where V_{k-1} is the linear space spanned by $\{x_1 - x, \cdots, x_{k-1} - x\}$.

Proof. Since $\Theta^{n-2}(\mu, y)$ is $H^{n-2}\lfloor\Sigma$ approximate continuous at x, we have by the definition that there is a positive function $\epsilon(r) > 0$ with $\lim_{r\downarrow 0} \epsilon(r) = 0$ such that

$$\begin{aligned} H^{n-2}\left(\left\{y \in \Sigma \cap B_r(x) : \left|\Theta^{n-2}(\mu, y) - \Theta^{n-2}(\mu, x)\right| \ge \epsilon(r)\right\}\right) &\le \frac{s(n)}{2} r^{n-2} \\ &< \frac{1}{2} r^{n-2}, \quad (4.20) \end{aligned}$$

where $s(n) > 0$ is a sufficiently small number to be determined later.
We want to show that there are $(n-2)$ points $\{x_1, \cdots, x_{n-2}\}$ in the set

$$\Sigma(x, r) \equiv \left\{y \in \Sigma \cap B_r(x) : |\Theta^{n-2}(\mu, y) - \Theta^{n-2}(\mu, x)| < \epsilon(r)\right\}$$

that satisfies the condition (ii).
Suppose that the above statement were false. Then $\Sigma(x, r)$ is contained in an sr-neighborhood of a $(n-3)$-dimensional plane $L_x \subset \mathbb{R}^n$ passing through x (note that $x \in \Sigma(x, r)$). In other words, for $r_i \downarrow 0$ if $y \in B_{r_i}(x) \cap \Sigma$ then one has *either*

$$|\Theta^{n-2}(\mu, y) - \Theta^{n-2}(\mu, x)| \ge \epsilon(r_i)$$

or y belongs to the sr_i-neighborhood of $L_i \cap B_{r_i}(x)$ for some $(n-3)$-dimensional plane $L_i \subset \mathbb{R}^n$ passing through x.
Now we want to estimate $\mu\left(B_{r_i}(x) \cap \Sigma\right)$. Note that for r_i small,

$$\mu\left(B_{r_i}(x) \cap \Sigma\right) \ge \frac{1}{2}\Theta^{n-2}\left(\mu, x\right) r_i^{n-2}. \quad (4.21)$$

On the other hand, by the upper semicontinuity of $\Theta^{n-2}(\mu, y)$, we have for r_i small,

$$\Theta^{n-2}(\mu, y) \le 2\Theta^{n-2}(\mu, x), \ \forall y \in B_{r_i}(x).$$

This implies

$$\Theta^{n-2}(\mu, y) \le 2\Theta^{n-2}(\mu, x), \ H^{n-2} \text{ for a.e. } y \in B_{r_i}(x) \cap \Sigma.$$

Hence

$$\begin{aligned} &\mu\left(\left\{y \in B_{r_i}(x) \cap \Sigma : |\Theta^{n-2}(\mu, y) - \Theta^{n-2}(\mu, x)| \ge \epsilon(r_i)\right\}\right) \\ &\le 2\Theta^{n-2}(\mu, x) H^{n-2}\left(\left\{y \in B_{r_i}(x) \cap \Sigma : |\Theta^{n-2}(\mu, y) - \Theta^{n-2}(\mu, x)| \ge \epsilon(r_i)\right\}\right) \\ &\le 2\Theta^{n-2}(\mu, x)\frac{s(n)}{2} r_i^{n-2} = s(n)\Theta^{n-2}(\mu, x) r_i^{n}. \quad (4.22) \end{aligned}$$

Note that we may cover sr_i-neighborhood of $L_i \cap B_{r_i}(x)$ by $C(n)s^{3-n}$ balls of radius of sr_i, since L_i is an $(n-3)$-dimensional plane through x. Let $\{B_{sr_i}(y_j)\}_{j=1}^{l}$, $y_j \in B_{r_i}(x)$, be such a cover with $l = C(n)s^{3-n}$. Then

$$\mu\left(sr_i - \text{neighborhood of } L_i \cap B_{r_i}(x)\right) \leq \sum_{j=1}^{l} \mu\left(B_{sr_i}(y_j)\right).$$

To estimate the right hand side, observe that there is $r_x > 0$ such that

$$\mu(B_r(r)) \leq \frac{3}{2}\Theta^{n-2}(\mu, x)r^{n-2}, \ \forall 0 < r \leq r_x.$$

Therefore, for $0 < r_i << r = r_x$, we have

$$\begin{aligned}
\mu\left(B_{sr_i}(y_j)\right) &= (sr_i)^{n-2}\frac{\mu\left(B_{sr_i}(y_j)\right)}{(sr_i)^{n-2}} \\
&\leq (sr_i)^{n-2}\frac{\mu\left(B_{r-r_i}(y_j)\right)}{(r-r_i)^{n-2}} \\
&\leq (sr_i)^{n-2}\left(\frac{r}{r-r_i}\right)^{n-2}\frac{\mu\left(B_r(x)\right))}{r^{n-2}} \\
&\leq 2(sr_i)^{n-2}\Theta^{n-2}(\mu, x).
\end{aligned}$$

Therefore we obtain

$$\begin{aligned}
\mu\left(sr_i - \text{neighborhood of } L_i \cap B_{r_i}(x)\right) &\leq C(n)s^{3-n}2(sr_i)^{n-2}\Theta^{n-2}(\mu, x) \\
&= 2C(n)s(n)r_i^{n-2}\Theta^{n-2}(\mu, x). \quad (4.23)
\end{aligned}$$

Combining (4.22) with (4.23), we obtain

$$\mu(B_{r_i}(x) \cap \Sigma) \leq s(n)\left(1 + 2C(n)\right)\Theta^{n-2}(\mu, x)r_i^{n-2} < \frac{1}{2}\Theta^{n-2}(\mu, x)r_i^{n-2}$$

if we choose $s(n) < \frac{1}{2(2C(n)+1)}$. This contradicts (4.21) and complete the proof. □

By Federer's structure theorem ([55] §3), it suffices to show that if $E \subset \Sigma$ is $(n-2)$-purely unrectifiable subset (i.e. if $F \subset \mathbb{R}^n$ is a $(n-2)$-rectifiable set then $H^{n-2}(E \cap F) = 0$), then $H^{n-2}(E) = 0$. We divide the proof into several lemmas.

Lemma 4.2.7 *For H^{n-2} a.e. $x \in \Sigma$ and for $\delta > 0$ there is $r_x > 0$ such that if $0 < r < r_x$ then there exists a $(n-2)$-plane $L = L(x, r) \subset \mathbb{R}^n$ through x so that*

$$\nu\left(B_r(x) \setminus L_{\delta r}\right) = 0,$$

where $L_{\delta r}$ is the δr-neighborhood of L in $\mathbb{R}^n$. As a consequence, for any $\delta_1, \delta_2 \in (0, 1)$ there are $r_0 > 0$ and $E_0 \subset \Sigma$ so that
(a) $H^{n-2}(\Sigma \setminus E_0) < \delta_1$.
(b) For $x \in E_0$ and $0 < r < r_0$, there is a $(n-2)$-plane $L = L(x, r) \subset \mathbb{R}^n$ through x such that $\nu(B_r(x) \setminus L_{\delta_2 r}) = 0$.

Proof. Note that for H^{n-2} a.e. $x \in \Sigma$, $\Theta^{n-2}(u,x) = 0$, $\Theta^{n-2}(\mu,x) \geq \epsilon_0^2$ and $\Theta^{n-2}(\mu,y)$ is H^{n-2}-approximate continuous at x for $y \in \Sigma$. Suppose it was false for such a $x_0 \in \Sigma$. Then for some $\delta > 0$, there is $r_i \downarrow 0$ such that

$$\nu\left(B_{r_i}(x_0) \setminus L_{\delta r_i}\right) > 0, \ \forall i \geq 1,$$

for any $(n-2)$-plane $L \subset \mathbb{R}^n$ through x_0.
Applying Lemma 4.2.6, there are $(n-2)$ points $\{x_1^i, \cdots x_{n-2}^i\} \subset \Sigma \cap B_{r_i}(x_0)$ such that

$$\Theta^{n-2}(\mu, x_j^i) \geq \Theta^{n-2}(\mu, x_0) - \epsilon_i, \text{ with } \lim_{i\to\infty} \epsilon_i = 0$$

for all $i \geq 1$ and $1 \leq j \leq n-2$, and

$$|x_1^i| \geq s r_i, \ \operatorname{dist}(x_j^i, x_0 + L_{j-1}^i) \geq s r_i, \quad 2 \leq j \leq n-2,$$

where

$$L_{j-1}^i = \operatorname{span}\left\{x_1^i - x_0, \cdots, x_{j-1}^i - x_0\right\}.$$

Let

$$\xi_j^i = \frac{x_j^i - x_0}{r_i}, \ 1 \leq j \leq n-2, \quad \text{and } \mu_i = \mu_{x_0,r_i} = |\nabla u_{x_0,r_i}|^2 \, dx + \nu_{x_0,r_i}.$$

Then, after taking possible subsequences, we can assume that there exist $\xi_j \in \mathbb{R}^n$ for $1 \leq j \leq n-2$ and two nonnegative Radon measures μ_* and ν_* such that as $i \to \infty$,

$$\xi_j^i \to \xi_j \text{ for } 1 \leq j \leq n-2, \ \mu_i \to \mu_* \text{ and } \ \nu_i \to \nu_*$$

Note that $\nu_* = \mu_*$, since $\Theta^{n-2}(u, x_0) = 0$. It is also easy to see that $\xi_j \in \pi(\mu_*)$ for $1 \leq j \leq n-2$. By the upper semicontinuity of $\Theta^{n-2}(\mu, x)$, we have

$$\Theta^{n-2}(\mu_*, \xi_j) \geq \limsup_{i\to\infty} \Theta^{n-2}(\mu_{x_0,r_i}, \xi_j^i) = \limsup_{i\to\infty} \Theta^{n-2}(\mu, x_j^i) \geq \Theta^{n-2}(\mu, x_0)$$

for $1 \leq j \leq n-2$. On the other hand, since μ_* is a tangent measure of μ at x_0, it follows from the discussion of the previous section that μ_* is a cone measure and

$$\Theta^{n-2}(\mu_*, x) \leq \Theta^{n-2}(\mu_*, 0) \text{ for all } x \in \mathbb{R}^n.$$

Therefore we have

$$L_{\mu_*}\left(= \left\{x \in \mathbb{R}^n \mid \Theta^{n-2}(\mu_*, x) = \Theta^{n-2}(\mu_*, 0)\right\}\right) = L^*,$$

where

$$L^* = \operatorname{span}\{\xi_1, \cdots, \xi_{n-2}\} \text{ is a } (n-2)\text{-plane in } \mathbb{R}^n.$$

This, combined with Lemma 4.1.4 (v), implies that there is a $\delta > 0$ such that for i sufficiently large

$$\nu_i\left(B_1 \setminus (L^*)_\delta\right) = 0.$$

This contradicts the choice of μ_i. Thus the first part of Lemma 4.2.7 holds.

The second part follows from the first part easily. In fact, for any $k, l \geq 1$, let Σ_{kl} be the subset of Σ containing $x \in \Sigma$ such that

$$\nu\left(B_r(x) \setminus L_{l^{-1}r}\right) = 0, \ \forall 0 < r < k^{-1}$$

for some $(n-2)$-plane $L \subset \mathbb{R}^n$ through x. Then we have

$$H^{n-2}\left(\Sigma \setminus \left(\cup_{k=1}^{\infty}\left(\cup_{l=1}^{\infty}\Sigma_{kl}\right)\right)\right) = 0.$$

This clearly implies that there is $l = l(k) \to \infty$ so that

$$\lim_{k\to\infty} H^{n-2}\left(\Sigma \setminus \Sigma_{kl(k)}\right) = 0$$

and hence the conclusion of the second part also holds by choosing $E_0 = \Sigma_{kl(k)}$ for k sufficiently large. □

Lemma 4.2.8 *If $E \subset \pi(\mu)$ is a purely $(n-2)$-unrectifiable set for some $\mu \in \mathcal{M}$, then*

$$H^{n-2}\left(P_L(E)\right) = 0$$

for any $(n-2)$-plane $L \subset \mathbb{R}^n$, where P_L is the orthogonal projection of $\mathbb{R}^n$ onto L.

Proof. We may assume $H^{n-2}(E) > 0$. For $\epsilon \in (0, \frac{1}{2})$, by Lemma 4.2.7 there are $E_0 \subset E$ and $r_0 > 0$ such that
(a) $H^{n-2}(E \setminus E_0) < \epsilon$.
(b) For $x \in E_0$ and $0 < r < r_0$, there is a $(n-2)$-plane $L \subset \mathbb{R}^n$ through x such that

$$E_0 \cap \left(B_r(x) \setminus L_{\epsilon r}\right) = \emptyset,$$

and
(c)

$$\mu\left(E \cap B_r(x)\right) \geq \frac{\Theta^{n-2}(\mu, x)}{2} r^{n-2} \geq \frac{\epsilon_0^2}{2} r^{n-2}.$$

Since E is purely unrectifiable, it follows from the characterization of rectifiable sets (cf. [55] §3.3.5) that for H^{n-2} a.e. $x \in E_0$, there is $y \in E_0 \cap B_r(x) \subset (B_r(x) \cap L_{\epsilon r})$ such that

$$|P_L\left(y - x\right)| \leq \frac{\epsilon}{4}|y - x|.$$

Therefore we have

$$H^{n-2}\left(P_L(B_r(x) \cap E_0)\right) \leq H^{n-2}\left(P_L(B_r(x) \cap L_{\epsilon r})\right) \leq 4\epsilon r^{n-2}. \tag{4.24}$$

By (c) and the Vitali's covering Lemma ([55] §2.8.15), we can almost cover E_0 by disjoint balls $\{B_{r_j}(x_j)\}$ with $x_j \in E_0$ such that both (c) and (4.24) are valid. Hence

$$\begin{aligned} H^{n-2}\left(P_L(E_0)\right) &\leq \sum_{j=1}^{\infty} H^{n-2}\left(P_L(E_0 \cap B_{r_j}(x_j))\right) \\ &\leq 4\epsilon \sum_{j=1}^{\infty} r_j^{n-2} \leq \frac{8\epsilon}{\epsilon_0^2} \sum_{j=1}^{\infty} \mu\left(E_0 \cap B_{r_j}(x_j)\right) \leq \frac{8\epsilon}{\epsilon_0^2}\mu(E). \end{aligned}$$

On the other hand

$$H^{n-2}\left(P_L(E \setminus E_0)\right) \le H^{n-2}(E \setminus E_0) \le \epsilon.$$

Thus we have

$$H^{n-2}(P_L(E)) \le \epsilon\left(1 + \frac{8\mu(E)}{\epsilon_0^2}\right).$$

This implies the conclusion, since $\epsilon > 0$ is arbitrary. □

Lemma 4.2.9 *If $\mu = |\nabla u|^2\,dx + \nu \in \mathcal{M}$, $\pi(\mu) = \Sigma$ and $E \subset \Sigma$ with $H^{n-2}(E) > 0$, then*

$$\lim_{r\downarrow 0} \sup_{L\in G_{n-2}(n)+\{x\}} \frac{H^{n-2}\left(P_L(E\cap B_r(x))\right)}{\alpha(n-2)r^{n-2}} \ge \frac{1}{2} \quad \text{for } H^{n-2} \text{ a.e. } x \in E. \tag{4.25}$$

Proof. As in Lemma 4.2.7, let $x_0 \in E$ such that $\Theta^{n-2}(u, x_0) = 0$, $\Theta^{n-2}(\mu, x_0) \ge \epsilon_0^2$, and $\Theta^{n-2}(\mu, \cdot)$ is H^{n-2}-approximate continuous at x_0 on E. Suppose this lemma was false at x_0. Then there exist $r_i \downarrow 0$ such that

$$\lim_{r_i\downarrow 0} \sup_{L\in G_{n-2}(n)+\{x\}} \frac{H^{n-2}\left(P_L(E\cap B_{r_i}(x_0))\right)}{\alpha(n-2)r_i^{n-2}} < \frac{1}{2}. \tag{4.26}$$

By passing to subsequences, we assume that

$$|\nabla u_{x_0,r_i}|^2\,dx \to 0,\ \mu_{x_0,r_i} \to \mu_*,\ \nu_{x_0,r_i} \to \nu_*.$$

Moreover, we have

$$\mu_* = \nu_* = \Theta^{n-2}(\mu, x_0)H^{n-2}\lfloor\left(\mathbb{R}^{n-2} \times \{(0,0)\}\right).$$

For $i \ge 1$, there are sequences of stationary harmonic maps $\{u_{ij}\}_{j=1}^{\infty}$ in B_2 such that

$$|\nabla u_{ij}|^2\,dx \to \nu_{x_0,r_i} + |\nabla u_{x_0,r_i}|^2\,dx \ \text{ as } j \to \infty.$$

Define

$$\alpha_{ij} \equiv \sum_{k=1}^{n-2} \left|\frac{\partial u_{ij}}{\partial x_k}\right|^2 dx.$$

Then, as in the proof of Lemma 4.2.7, we can find a diagonal subsequence $j = j(i)$ such that

$$|\nabla u_{ij(i)}|^2\,dx \to \mu_* \ \text{ as } i \to \infty.$$

Hence for any $\delta > 0$ there is a sufficiently large i_0 such that for $i \ge i_0$

$$\alpha_{ij}(B_{\frac{3}{2}}) \le \delta. \tag{4.27}$$

Now we define $F_{ij} : (\mathbb{R}^{n-2} \times \{(0,0)\}) \times (0,1) \to \mathbb{R}_+$ by

$$F_{ij}(a,\epsilon) = \int_{B_2^n} |\nabla u_{ij}|^2\,(a+x)\psi_\epsilon(x_1)\phi^2(x_2)\,dx \tag{4.28}$$

where $x = (x_1, x_2) \in \mathbb{R}^{n-2} \times \mathbb{R}^2$, $\phi \in C_0^\infty(B_2^2)$, $\psi_\epsilon(x_1) = \epsilon^{2-n}\psi(\frac{x_1}{\epsilon})$ with $0 \le \psi \in C_0^\infty(B_1^{n-2})$ and

$$\int_{\mathbb{R}^{n-2}} \psi(x_1)\, dx_1 = 1.$$

Direct calculations imply, for $1 \le k \le n-2$,

$$\begin{aligned} \frac{\partial F_{ij}}{\partial a_k} &= 2\sum_{l=1}^{n-2} \frac{\partial}{\partial a_l} \int_{B_2^n} \left(\frac{\partial u_{ij}}{\partial x_l} \frac{\partial u_{ij}}{\partial x_k} \right) (a+x)\psi_\epsilon(x_1)\phi^2(x_2)\, dx \\ &\quad -2 \sum_{l=n-1}^{n} \int_{B_2^n} \left(\frac{\partial u_{ij}}{\partial x_l} \frac{\partial u_{ij}}{\partial x_k} \right) (a+x)\psi_\epsilon(x_1) \frac{\partial \phi^2}{\partial x_l}\, dx. \end{aligned} \tag{4.29}$$

Now we recall Allard's strong constancy lemma (cf. [1]).

Lemma 4.2.10 *Suppose F, f and G are smooth on B_1^{n-2} and satisfy*

$$\nabla F = f + divG \text{ in } B_1^{n-2}, \tag{4.30}$$

and

$$\|f\|_{L^1(B_1^{n-2})} + \|G\|_{L^1(B_1^{n-2})} \le \delta. \tag{4.31}$$

Then for any $\delta_1 > 0$ there is $\delta_0 > 0$ depending on δ_1 and $\|F\|_{L^1(B_1)}$ such that

$$\|F - c\|_{L^1(B_1)} \le \delta_1 \text{ whenever } \delta \le \delta_0. \tag{4.32}$$

Note that if we denote $F^\epsilon = F_{ij}(a, \epsilon)$, the second term of (4.29) by f_k^ϵ ($1 \le k \le n-2$) and

$$G_{kl}^\epsilon = \int_{B_2^n} \left(\frac{\partial u_{ij}}{\partial x_l} \frac{\partial u_{ij}}{\partial x_k} \right) (a+x)\psi_\epsilon(x_1)\phi^2(x_2)\, dx.$$

Then we have

$$\nabla F^\epsilon = f^\epsilon + \operatorname{div}(G^\epsilon) \text{ in } B_2^{n-2},$$

and since $\alpha_{ij(i)} \to 0$, we have for any $\delta > 0$

$$\|f^\epsilon\|_{L^1(B_2^{n-2})} + \|G^\epsilon\|_{L^1(B_2^{n-2})} \le \delta$$

holds for $u_{ij(i)}$ with $i \ge i_0$ sufficiently large. Therefore by Lemma 4.2.10 we conclude that for any $\delta_1 > 0$

$$\|F_{ij}(a, \epsilon) - C_{ij}(\epsilon)\|_{L^1(B_{\frac{3}{2}})} \le \delta_1 \tag{4.33}$$

holds for i sufficiently large and $j = j(i)$ for some positive $C_{ij}(\epsilon)$.
Taking $\epsilon \to 0$, (4.33) gives that for i sufficiently large

$$\|F_{ij}(a) - C_{ij}\|_{L^1(B_{\frac{3}{2}})} \le \delta_1 \tag{4.34}$$

holds for $j = j(i)$ for some $C_{ij} > 0$, where

$$F_{ij}(a) = \int_{B_2^2} |\nabla u_{ij}|^2 (a, x_2)\phi^2(x_2)\, dx_2.$$

Choosing ϕ to be a cut-off function of $B^2_{\frac{1}{2}}$, we can show that

$$\lim_{i\to\infty} C_{ij(i)} = \Theta^{n-2}(\mu, x_0).$$

Let σ_i be the projection of ν_{x_0,r_i} on $\mathbb{R}^{n-2}\times\{(0,0)\}$. Then, by a variant of the Fubini's theorem on slice measures (cf. Evans [46]), we conclude that for any fixed i,

$$F_{ij}(x_1)\,dx \to \sigma_i + \epsilon_i(x_1)\,dx_1$$

as convergence of Radon measures on B^{n-2}_1 as $j\to\infty$, where

$$\epsilon_i(x_1) = \int_{B^2_2} |\nabla u_{x_0,r_i}|^2(x_1,x_2)\phi^2(x_2)\,dx_2.$$

Assume $C_i = \lim_{j\to\infty} C_{ij}$ exists. Then by (4.34) we have

$$\sigma_i(x_1) = C_i dx_1 - \epsilon_i(x_1)\,dx_1 + \beta_i(x_1) \tag{4.35}$$

where β_i is a signed Radon measure with its total mass on $B_{\frac{3}{2}}$ bounded by δ. By choosing $\delta > 0$ sufficiently small and i sufficiently large, we conclude that

$$\sigma_i(B^{n-2}_1) \geq \frac{\Theta^{n-2}(\mu, x_0)}{4}.$$

Hence $P_{\mathbb{R}^{n-2}\times\{(0,0)\}}\,(\mathrm{supp}(\nu_{x_0,r_i}\cap B_1))$ contains at least half of B^{n-2}_1. That is, for i sufficiently large,

$$\frac{H^{n-2}\left(P_{\mathbb{R}^{n-2}\times\{(0,0)\}}(E\cap B_{r_i}(x_0)\right)}{\alpha(n-2)r_i^{n-2}} \geq \frac{1}{2}.$$

This contradicts (4.26). Hence the proof is complete. □

Proof of Theorem 4.2.5: Suppose that $\Sigma = \pi(\mu)$, $\mu\in\mathcal{M}$, is not $(n-2)$-rectifiable. Then there exists a purely $(n-2)$-unrectifiable subset $E\subset\Sigma$ with $H^{n-2}(E) > 0$. But this is impossible according to Lemma 4.2.8 and Lemma 4.2.9. □

4.3 Strong convergence and interior gradient estimates

In this section, we will show the obstruction for both strong convergence in H_Λ and the interior gradient estimates of stationary harmonic maps.

We begin with

Definition 4.3.1 For $k = 2,\cdots,n-1$, a smooth map $\omega\in C^\infty(S^k, N)$ is called a harmonic S^k if it is a nonconstant harmonic map.

The main theorem of this section is

Theorem 4.3.2 *Assume that N does not admit any harmonic S^k for $k = 2,\cdots,n-1$. Let M be a n-dimensional compact Riemannian manifold without boundary and*

$u \in H^1(M,N)$ be a stationary harmonic map, then $u \in C^\infty(M,N)$ and the following gradient estimate holds:

$$\|\nabla u\|_{L^\infty(M)} \le C(M,N,E) \tag{4.36}$$

where $E = \int_M |\nabla u|^2$.

As an immediate corollary, we have

Corollary 4.3.3 *If the universal cover $\tilde{N}$ of N supports a strictly convex function, then any stationary harmonic map $u \in H^1(M,N)$ is smooth and satisifes (4.36).*

For any sequence $\{u_i\} \subset H_\Lambda$, let $u \in H^1(\Omega, N)$ be such that $u_i \to u$ weakly in $H^1(\Omega, N)$ and

$$\mu_i = |\nabla u_i|^2 \, dx \to \mu = |\nabla u|^2 \, dx + \nu$$

as convergence of Radon measures for some *defect* measure ν in Ω. From the discussion of previous sections, we know that

$$u_i \to u \text{ strongly in } H^1_{\text{loc}} \Leftrightarrow \nu = 0 \Leftrightarrow H^{n-2}(\Sigma) = 0 \quad \text{where } \Sigma = \pi(\mu). \tag{4.37}$$

The following lemma is the key to prove the interior gradient estimate for stationary harmonic maps.

Lemma 4.3.4 *Suppose that for some $\mu \in \mathcal{M}$, $\Sigma = \pi(\mu)$ has positive H^{n-2}-measure. Then there exists at least one harmonic S^2 in N.*

Proof. Since $H^{n-2}(\Sigma) > 0$, it follows from Proposition 1.1.5 that $\Sigma_{n-2} \setminus \Sigma_{n-3} \neq \emptyset$ and hence there exist $x_0 \in \Sigma_{n-2}$ and a tangent measure $\mu_* \in T_{x_0}\mu$ $(\subset \mathcal{M})$ such that $\Sigma_* = \pi(\mu_*)$ satisfies

$$\Sigma_* = \mathbb{R}^{n-2} \times \{(0,0)\}\,, \;\; \mu_* = \Theta^{n-2}(\mu, x_0)\, H^{n-2} \lfloor \Sigma_*. \tag{4.38}$$

Let $\{v_i\} \subset H_\Lambda$ $(\Lambda \ge \Theta^{n-2}(\mu, x_0) > 0)$ be such that $v_i \to$ constant weakly in $H^1(B_1^n, N)$ and in $C^2\left(B_1^n \setminus (B_1^{n-2} \times \{(0,0)\}), N\right)$,

$$\eta_i = |\nabla v_i|^2 \, dx \to \mu_*$$

as convergence of Radon measures in B_1^n, and

$$\lim_{i\to\infty} \sum_{\alpha=1}^{n-2} \int_{B_1^n} |\nabla_\alpha v_i|^2 = 0. \tag{4.39}$$

Set $X_1 = (x_1, \cdots, x_{n-2})$ and $X_2 = (x_{n-1}, x_n)$. For $i \ge 1$, define $f_i : B_{\frac{1}{2}}^{n-2} \to \mathbb{R}$ by

$$f_i(X_1) = \sum_{\alpha=1}^{n-2} \int_{B_1^2} |\nabla_\alpha v_i|^2 (X_1, X_2)\, dX_2.$$

Then Fubini's theorem implies

$$\lim_{i\to\infty} \|f_i\|_{L^1(B_{\frac{1}{2}}^{n-2})} = 0.$$

Define the (local) Hardy-Littlewood maximal function (cf. [192])

$$M(f_i)(X_1) = \sup_{0<r\le\frac12} r^{2-n}\int_{B_r(X_1)} f_i, \quad \text{for } X_1 \in B^{n-2}_{\frac12}.$$

By the partial regularity theorem of stationary harmonic maps and the weak L^1-estimate for the Hardy-Littlewood maximal functions, we conclude that there are $\{X_1^i\} \in B^{n-2}_{\frac12}$ such that

$$v_i \text{ is smooth near } \{X_1^i\}\times B^2_{\frac12}\ \forall i\ge 1, \tag{4.40}$$

and

$$\lim_{i\to\infty} M(f_i)(X_1^i) = 0. \tag{4.41}$$

For i sufficiently large, there exist

$$0<\delta_i\to 0, \text{ and } X_2^i \in B^2_{\frac14} \text{ with } X_2^i \to (0,0)$$

such that

$$\begin{aligned}&\delta_i^{2-n}\int_{B^{n-2}_{\delta_i}(X_1^i)\times B^2_{\delta_i}(X_2^i)} |\nabla v_i|^2 \\ &= \max_{X_2\in B^2_{\frac12}} \delta_i^{2-n}\int_{B^{n-2}_{\delta_i}(X_1^i)\times B^2_{\delta_i}(X_2)} |\nabla v_i|^2 = \frac{\epsilon_0^2}{c(n)}\end{aligned} \tag{4.42}$$

where $c(n)>0$ is a large constant to be chosen later.
To see (4.42), we observe that since u_i is smooth near $\{X_1^i\}\times B^2_{\frac12}$ we have that for any $i\ge 1$ fixed,

$$g_i(\delta) \equiv \max_{X_2\in B^2_{\frac12}} \delta^{2-n}\int_{B^{n-2}_{\delta}(X_1^i)\times B^2_{\delta}(X_2)} |\nabla v_i|^2 \to 0$$

as $\delta\downarrow 0$. On the other hand, for any fixed $\delta>0$, we must have

$$\lim_{i\to\infty} g_i(\delta) > \epsilon_0^2.$$

For, otherwise, there exists $\delta_0>0$ such that $g_i(\delta_0)\le\epsilon_0^2$ for $i\ge 1$ sufficiently large. Hence, by the small energy regularity theorem, we have

$$\delta_0 \max_{B^{n-2}_{\frac{\delta_0}{2}}\times B^2_{\frac{\delta_0}{2}}} |\nabla v_i| \le C\epsilon_0.$$

This contradicts $|\nabla v_i|^2\,dx \to \mu_*$. Therefore for i sufficiently large, there is $\delta_i\to 0$ such that (4.42) holds.
Suppose $X_2^i \not\to (0,0)$. Then we may assume that $X_2^i\to X_2^0$ for some $(0,0)\ne X_2^0\in B^2_{\frac12}$. Then by (3.42) we have, for $r_0 = |X_2^0|>0$,

$$\left(\frac{r_0}{2}\right)^{2-n}\int_{B_{\frac{r_0}{2}}(X_1^i,X_2^0)} |\nabla v_i|^2 \ge \frac{\epsilon_0^2}{c(n)}$$

for i sufficiently large. In particular,

$$\int_{B_1^{n-2}\times(B^2_{2r_0}\setminus B^2_{\frac{r_0}{2}})} |\nabla v_i|^2 \geq C(\epsilon_0, r_0) > 0$$

for sufficiently large i. This contradicts

$$v_i \to \text{constant in } C^2\left(B_1^n \setminus (B_1^{n-2}\times\{(0,0)\}), N\right).$$

Now we proceed the blow-up scheme as follows. Let $p_i = (X_1^i, X_2^i)$, $R_i = (4\delta_i)^{-1}$, and $\Omega_i = B^{n-2}_{R_i}\times B^2_{R_i}$. Since $R_i \to \infty$, Ω_i converges to $\mathbb{R}^n$. Define $w_i : \Omega_i \to N$ by

$$w_i(x) = v_i(p_i + \delta_i x),\ \ x \in \Omega_i.$$

Then w_i is a stationary harmonic map on Ω_i. Moreover, (4.41) and (4.42) imply

$$\lim_{i\to\infty}\sup_{0<R<R_i}\left\{R^{2-n}\int_{B_R^{n-2}\times B^2_{R_i}}\sum_{\alpha=1}^{n-2}|\nabla_\alpha w_i|^2\right\} = 0, \tag{4.43}$$

$$\begin{aligned}\int_{B_1^{n-2}\times B_1^2}|\nabla w_i|^2 &= \frac{\epsilon_0^2}{c(n)}\\ &= \max\left\{\int_{B_1^{n-2}\times B_1^2(X_2)}|\nabla w_i|^2 : X_2 \in B^2_{R_i-1}\right\},\end{aligned} \tag{4.44}$$

$$\sup_i\left\{\int_{B_R^{n-2}\times B_R^2}|\nabla w_i|^2\right\} \leq C(\Lambda)R^{n-2},\ \forall 0<R<R_i. \tag{4.45}$$

Let $\phi_1 \in C_0^\infty(B_1^{n-2})$ be such that $0 \leq \phi_1 \leq 1$ and $\phi_1 = 1$ on $B^{n-2}_{\frac{3}{4}}$. Let $\phi_2 \in C_0^\infty(B_1^2)$ be such that $0 \leq \phi_1 \leq 1$ and $\phi_1 = 1$ on $B^2_{\frac{1}{2}}$. Define $F_i : B_1^{n-2}\times B^2_{R_i-1} \to \mathbb{R}$ by

$$F_i(a) = \int_{B_1^{n-2}\times B_1^2}|\nabla w_i|^2(a+x)\phi_1(X_1)\phi_2(X_2)\,dx.$$

Then direct calculations imply, for $k = 1, \cdots, n-2$,

$$\frac{\partial F_i}{\partial a_k} = -2\sum_{l=1}^n\int_{B_1^{n-2}\times B_1^2}\left\langle\frac{\partial w_i}{\partial y_l}, \frac{\partial w_i}{\partial y_k}\right\rangle(y+a)\frac{\partial}{\partial y_l}(\phi_1\phi_2)\,dy. \tag{4.46}$$

Thus (4.43) yields

$$\frac{\partial F_i}{\partial a_k} \to 0 \text{ uniformly for } a \in B_2^{n-2}\times B^2_{R_i-1},\ \text{as } i\to\infty$$

for $k = 1, \cdots, n-2$. Hence by (4.44) if we choose $c(n) = 2^{n+3}$, then

$$2^{2-n}\int_{B_2^{n-2}\times B_2^2}|\nabla w_i|^2(Y_1, Y_2+b)\,dY_1dY_2 \leq \epsilon_0^2,\ \text{for all } b \in B^2_{R_i-2}. \tag{4.47}$$

Therefore, by Theorem 3.3.8, we may assume that there exists a smooth harmonic map $w : B^{n-2}_{\frac{3}{2}} \times \mathbb{R}^2 \to N$ such that

$$w_i \to w \text{ in } C^2_{\text{loc}}\left(B^{n-2}_{\frac{3}{2}} \times \mathbb{R}^2, N\right) \quad \text{as } i \to \infty.$$

It follows from (4.43) that

$$\sum_{\alpha=1}^{n-2} \int_{B^{n-2}_{\frac{3}{2}} \times \mathbb{R}^2} |\nabla_\alpha w|^2 = 0$$

so that $w(Y_1, Y_2) = w(Y_2)$ is independent of the first $(n-2)$-variables. Hence $w : \mathbb{R}^2 \to N$ is a smooth harmonic map. This and (4.44) and (4.44) imply that

$$\frac{\epsilon_0^2}{2^{n+3}} \le \int_{\mathbb{R}^2} |\nabla w|^2 \le \Lambda.$$

Therefore w can be lifted to be a nontrivial harmonic map from S^2 to N by the removability theorem of isolated singularity (see Sacks-Uhlenbeck [164] or §6 below). This completes the proof. □

Proof of Theorem 4.3.2:
Since N doesn't support any harmonic S^2, it follows that the set E_Λ of stationary harmonic maps from M to N whose energies are bounded by Λ is sequentially compact in $H^1(M, N)$. In fact, if $\{u_i\} \subset E_\Lambda$ then we can assume $u_i \to u$ weakly in $H^1(M, N)$. Let Σ be the concentration set associated with the sequence. Then lemma 2.3.3 implies that $\Sigma_{n-2} \setminus \Sigma_{n-3} = \emptyset$. Hence $H^{n-2}(\Sigma) = 0$ and $\dim_H(\Sigma) \le n-3$, and $u_i \to u$ strongly in $H^1(M, N)$. Now we can apply the Federer dimension reduction argument as in §2.3 to conclude that $\Sigma_j = \emptyset$ for all $0 = j, \cdots, n-3$, since there is no harmonic S^k for $k = 3, \cdots, n-1$. Therefore $\Sigma = \emptyset$ and $u_i \to u$ in $C^2(M, N)$. In particular any stationary harmonic map from M to N is smooth. To see the estimate (4.36), we argue by contradiction. Suppose that there are smooth stationary harmonic maps $v_i : M \to N$ such that

$$\|\nabla v_i\|_{C^0(M)} \to +\infty, \text{ but } \sup_i \int_M |\nabla v_i|^2 \le C < +\infty.$$

Then $\{v_i\} \subset H^1(M, N)$ is not compact in $H^1(M, N)$, which is impossible. The proof is complete. □

Proof of Corollary 4.3.3:
It suffices to prove that there is no harmonic S^k in N for $k = 2, \cdots, n-1$. Let $\omega \in C^\infty(S^k, N)$ be any harmonic map. Let $\Pi : \tilde{N} \to N$ be the covering map. Then it is well-known that there exists a smooth harmonic map $\tilde{\omega} : S^k \to \tilde{N}$ such that $\omega = \Pi \circ \tilde{\omega}$.

Let $\rho : \tilde{N} \to \mathbb{R}$ be a strictly convex function so that there is $c_N > 0$ such that

$$\nabla^2 \rho(y)(\xi, \xi) \ge c_N |\xi|^2 \text{ for any } y \in N \text{ and } \xi \in T_y N.$$

Then by the chain rule (cf. [102]) we have

$$\Delta(\rho \circ \tilde{\omega}) = \mathrm{tr}\left(\nabla^2 \rho(\tilde{\omega})(\nabla\tilde{\omega}, \nabla\tilde{\omega})\right) \geq c_N |\nabla\tilde{\omega}|^2. \tag{4.48}$$

This implies that $\rho \circ \tilde{\omega}$ is a subharmonic function on S^k. Hence, by the maximum principle, we conclude that $\rho \circ \tilde{\omega}$ is constant. This yields $|\nabla\tilde{\omega}| \equiv 0$ and $\tilde{\omega}$ is a constant. Therefore $\omega = \Pi \circ \tilde{\omega}$ is also a constant. □

4.4 Boundary gradient estimates

In this section we will address the boundary gradient estimates for smooth harmonic maps. We would like to remark that the boundary monotonicity inequality similar to (2.42) doesn't seem to follow from the stationary identity (3.41) in which the variational vector field X is required to be in $C_0^\infty(\Omega, \mathbb{R}^n)$. Hence if we consider the class of stationary harmonic maps, then we need to impose the boundary monotonicity inequality to analyze the boundary gradient estimate. In fact, Wang [205] has proved a partial boundary regularity for stationary harmonic maps under the extra assumption that the boundary monotonicity inequality (4.49) holds.

For simplicity we assume $\Omega = B_1^+ \subset \mathbb{R}^n$ and $u \in C^\infty(\overline{B_1^+}, N)$ is a smooth harmonic map such that

$$E = \int_{B_1^+} |\nabla u|^2 \leq K < +\infty \ \text{ and } u|_\Gamma = \phi \text{ with } \|\phi\|_{C^2(\Gamma)} \leq K < +\infty$$

for some $K > 0$, where $\Gamma = \left\{x = (x', x_n) \in B_1^+ \mid x_n = 0\right\}$.

Before we analyze the behavior of smooth harmonic maps near boundary points, we need

Lemma 4.4.1 *There is a constant $\Lambda > 0$ depending only on n, K such that for any $x \in \Gamma$, $0 < \sigma \leq \rho$ with $B_\rho^+(x) \subset B_1^+$,*

$$\int_{B_\rho^+(x)\setminus B_\sigma^+(x)} |y - x|^{2-n} \left|\frac{\partial u}{\partial |y - x|}\right|^2 \leq E_\rho(x) - E_\sigma(x), \tag{4.49}$$

where

$$E_\rho(x) = e^{\Lambda\rho}\rho^{2-n} \int_{B_\rho^+(x)} |\nabla u|^2 + C(\Lambda)\rho.$$

Proof. Let ϕ be an extension of $\phi|_\Gamma$ to B_1^+ such that $\|\phi\|_{C^2(B_1^+)} \leq K$. For simplicity, assume $x = 0$. For $0 < r \leq 1$, multiply (1.8) by $x \cdot \nabla(u - \phi)(x)$ and integrate over B_r^+. By integration by parts, we have

$$\begin{aligned} &\left| r\int_{\partial B_r^+} |\nabla u|^2 - 2r\int_{\partial B_r^+} |\frac{\partial u}{\partial r}|^2 - (n-2)\int_{B_r^+} |\nabla u|^2 \right| \\ \leq\ & C(K)\left[\int_{\partial B_r^+} \left(r|\nabla u|^2 + |\nabla u| + r|\nabla u|\right) + r\int_{\partial B_r^+} |\frac{\partial u}{\partial r}|\right]. \end{aligned}$$

It is easy to see that this yields (4.49). □

It follows from (4.49) that $E_r(x)$ is a monotonically increasing function with respect to r for any $x \in \Gamma$. In particular,

$$\Theta^{n-2}(u,x) = \lim_{r \downarrow 0} \left(e^{\Lambda r} r^{2-n} \int_{B_r^+(x)} |\nabla u|^2 \right)$$

exists for any $x \in \Gamma$.

We also need the following small energy regularity at the boundary. For smooth harmonic maps, the proof can be found in [31]. For stationary harmonic maps satisfying (4.49), it was proved by [205].

Lemma 4.4.2 *There exists $\epsilon_0 = \epsilon(n,N,K) > 0$ such that if $E_r(x) \le \epsilon^2$ for $x \in \Gamma$ and $r > 0$, then*

$$r \max_{B^+_{\frac{r}{2}}(x)} |\nabla u| \le C\left(\epsilon_0, n, N, K\right). \tag{4.50}$$

We now are ready to prove the main theorem of this section (cf. [123] Page 822-824).

Theorem 4.4.3 *Assume that N does not support any harmonic S^2. Let $u \in C^\infty(\overline{B_1^+}, N)$ be a harmonic map on B_1^+ given as above. Then there exists $\delta_0 = \delta(K,n,N) > 0$ such that*

$$|\nabla u|(x) \le C\left(n,N,K\right) \text{ for all } x \in \left\{ x = (x', x_n) \in B^+_{\frac{1}{2}} \mid 0 \le x_n \le \delta_0 \right\}. \tag{4.51}$$

Proof. We argue by contradiction. Suppose that the theorem were false. Then for any $\delta > 0$ and $K > 0$, there exist $\{u_i\} \subset C^\infty(\overline{B_1^+}, N)$ with $u_i|_\Gamma = \phi_i$,

$$\|\phi\|_{C^2(\Gamma)} \le K, \quad \int_{B_1^+} |\nabla u_i|^2 \le K,$$

but

$$\sup_{B^+_{\frac{1}{2}} \cap \Gamma_\delta} |\nabla u_i| \to \infty, \text{ where } \Gamma_\delta = \left\{ x = (x', x_n) \in B_1^+ : 0 \le x_n \le \delta \right\}.$$

In particular, $u_i \to u$ weakly in $H^1(B_1^+, N)$ but not strongly in $H^1(B_1^+, N)$. Hence there exists a nonnegative Radon measure ν on Γ_δ with $\nu(\Gamma_\delta) > 0$ such that

$$|\nabla u_i|^2 \, dx \to \mu \equiv |\nabla u|^2 \, dx + \nu$$

as convergence of Radon measures on Γ_δ. We may assume that there is $\phi \in C^2(\Gamma, N)$ such that $\phi_i \to \phi$ in $C^{1,\alpha}(\Gamma)$ for any $0 < \alpha < 1$. As in §4.1 and §4.2, we let $\mathcal{M}_+$ denote all Radon measures μ on Γ_δ obtained as above and $\Pi(\mu) = \Sigma$, where Σ is the energy concentration set in B_1^+, associated with the sequences.

Write $\Sigma = \Sigma_1 \cap \Sigma_2$, with $\Sigma_1 = \Sigma \cap \Gamma$ and $\Sigma_2 = \Sigma \setminus \Sigma_1$. Since $\nu(\Gamma_\delta \cap B_1^+) > 0$, we must have $H^{n-2}(\Sigma) > 0$. On the other hand, since there is no harmonic S^2 in

N, we already know that $H^{n-2}(\Sigma_2) = 0$. Hence $H^{n-2}(\Sigma_1) > 0$. Now we can modify the argument in §4.1 and §4.2 to conclude that $\Sigma_1 \subset \Gamma$ is a $(n-2)$-rectifiable set. Moreover, for H^{n-2} a.e. $x_0 \in \Sigma_0$, there exists $r_i \downarrow 0$ and a $(n-2)$-dimensional plane $L \subset \partial\mathbb{R}^n_+$ such that

$$\nu_{x_0, r_i} \to \nu_*, \quad \text{and } \nu_* = \Theta_0 H^{n-2} \lfloor L \text{ for some } \Theta_0 > 0. \tag{4.52}$$

In particular, by a diagonal process, we may assume that there are $\{v_i\}$ of smooth harmonic maps in B_1^+, with $v_i|_\Gamma \to \text{const}$, so that

$$|\nabla v_i|^2 \, dx \to \nu_* = \Theta_0 H^{n-2} \lfloor L, \int_{B_1^+} |\nabla_y v_i|^2 \to 0 \text{ for all } y \in T. \tag{4.53}$$

Now we can follow the blow up scheme in §4.3 to conclude that there exist either a nonconstant harmonic map from S^2 to N or a harmonic map from $\mathbb{R}^2_+$ to N with finite energy and being constant on $\partial\mathbb{R}^2_+$. The latter is impossible by a well known theorem of Lemaire [114]. Hence there exists at least a nontrivial harmonic S^2 in N, which contradicts our assumption.

The above argument shows that $u_i \to u$ strongly in $H^1(B_1^+, N)$. Now we can apply Federer's dimension reduction argument to show if $\Sigma \neq \emptyset$, then there would exist a $3 \le l \le n-1$ and a non trivial harmonic map $\phi \in C^\infty(S^l_+, N)$ with $\phi|_{\partial S^l_+} =$ const, where S^l_+ is the l-dimensional upper hemi sphere. But this is again impossible by the following theorem due to Schoen-Uhlenbeck ([172] page 263). Namely,

Lemma 4.4.4 *A smooth harmonic map $u_0 : S^l_+ \to N$ taking constant value on ∂S^l_+ is itself constant for $l \ge 2$.*

Proof. The result for $l = 2$ is proven by Lemaire [114]. For $n \ge 3$, one can perform a radial deformation on S^l_+ with center at the north pole as a domain variation of u. Since smooth harmonic maps are also critical points with respect to domain variations. Hence the first variation vanishes. On the other hand, direct computations show that the first variation is equal to

$$(2-l) \int_{S^l_+} (1-\rho)\, e(u_0)\, (\rho, \xi) \, .$$

Hence $e(u_0) \equiv 0$ and u_0 is constant. Applying this lemma, we then conclude that $\Sigma = \emptyset$ and hence the original convergence is C^2. Thus we can obtain the uniform boundary gradient estimate (4.51). □

Remark 4.4.5 A crucial property used in the blow-up analysis technique presented in this Chapter is the energy monotonicity formula of the problem under considerations. In particular, similar techniques have been employed by Tian in his important work on higher dimensional Yang-Mills fileds [200].

Chapter 5

Heat flows to Riemannian manifolds of NPC

In this chapter, we will present the classical theorem by Eells-Sampson [49] on the existence of global smooth solutions to the heat flow of harmonic maps into compact Riemannian manifolds with nonpositive sectional curvatures.

5.1 Motivation

Given two Riemannian manifolds (M, g) and (N, h), a fundamental problem in geometric toplogy is to find harmonic maps representing any given free homotopy class $\alpha \in [M, N]$. For example, one asks *under what conditions, can*

$$c_\alpha := \inf \left\{ \int_M e(u)\, dv_g \mid u \in C^\infty(M, N), [u] = \alpha \in [M, N] \right\} \tag{5.1}$$

be achieved?

The answer to these questions is delicate. For example, the following theorem, due to Eells-Wood [52], indicates that the problem may not always have solutions.

Proposition 5.1.1 *Let* $\left(M = T^2 = S^1 \times S^1, g\right)$ *be the flat torus,* $N = (S^2, h)$ *be the standard sphere,* $\alpha \in \left[T^2, S^2\right]$ *be the homotopy class whose topological degree is* 1*. Then there doesn't exist any harmonic map in* α*.*

Let's also mention two more examples that illustrate non-existence of nontrivial harmonic maps with constant boundary values. The first one is due to Lemaire [114].

Proposition 5.1.2 *For any ball* $B \subset \mathbb{R}^2$*, if* $u \in C^\infty(B, N)$ *is a harmonic map such that* $u|_{\partial B} = \text{constant}$*, then* $u = \text{constant}$.

Proof. First, by conformal transformation and conformal invariance of harmonic maps in dimensions two, we may identify B with S^2_+, the upper half sphere. Extending u from S^2_+ to N by letting $u(x_1, x_2, x_3) = u(x_1, x_2, -x_3)$ for $x_3 \le 0$. Then one can verify $u : S^2 \to N$ is harmonic map. Hence it has zero Hopf differential. This implies $u : S^2 \to N$ is conformal, $u|_{\partial S^2_+} = \text{constant}$, so $u = \text{constant}$. □

Proposition 5.1.3 *For $n \geq 3$, let $B^n \subset \mathbb{R}^n$ be the unit ball. Then any harmonic map $u \in C^\infty(\overline{B^n}, N)$, with $u|_{\partial B^n} = \text{constant}$, is constant.*

Proof. By the monotonicity identity (3.42), we have

$$(2-n)\int_{B^n} |\nabla u|^2 + \int_{\partial B^n} |\nabla u|^2 = 2\int_{\partial B^n} |\frac{\partial u}{\partial r}|^2. \tag{5.2}$$

Since $u = \text{constant}$ on ∂B^n, this implies

$$\int_{\partial B^n} |\frac{\partial u}{\partial r}|^2 + (n-2)\int_{B^n} |\nabla u|^2 = 0.$$

Hence $u = \text{constant}$ in B^n. □

These two examples indicate that for (M, g) with nonempty boundary ∂M, if we consider harmonic map representing relative free homotopy classes, then it may have no solutions.

The direct method to attack the above homotopy problem also has its limitation, since the topological property may not be preserved under weak convergence in $W^{1,2}(M, N)$. For example, let $u_\lambda(z) = \lambda z : \overline{\mathbb{R}}^2 = S^2 \to \overline{\mathbb{R}}^2 = S^2$, $\lambda > 0$. Then $\int_{S^2} e(u_\lambda) = 8\pi$ for $\lambda > 0$, $\deg(u_\lambda; S^2) = 1$, and $u_\lambda \to (0,0,1)$ weakly in $W^{1,2}(S^2, S^2)$, as $\lambda \to \infty$. Hence both the topological information and the Dirichlet energy have lost in the limiting process.

To overcome these difficulties, Eells-Sampson in their 1964's poineering work [49] have proposed to study the corresponding evolution problem: for any $u_0 \in C^\infty(M, N)$, find $u : M \times \mathbb{R}_+ \to N$ that solves

$$\partial_t u - \Delta_g u = A(u)(\nabla u, \nabla u) \ \text{ in } M \times (0, +\infty) \tag{5.3}$$

$$u|_{t=0} = u_0. \tag{5.4}$$

The basic ideas of [49] are two fold. First, a continuous deformation $u(\cdot, t)$ of u_0 preserves the (given) homotopy class. Second, (5.3) is the negative L^2-gradient flow of the Dirichlet energy E, under which the energy is decreasing and hence one may find critical points of E (i.e., harmonic maps) along the flow.

5.2 Existence of short time smooth solutions

Eells-Sampson [49] have established the following existence theorem on short time smooth solutions to (5.3) and (5.4).

Theorem 5.2.1 *Suppose both (M, g) and $(N, h) \subset \mathbb{R}^L$ are compact without boundaries. Then for any $u_0 \in C^\infty(M, N)$ there exists a maximal time $0 < T_{max} = T_{max}(u_0, M, N) \leq \infty$ such that (5.3)-(5.4) admits a unique, smooth solution $u \in C^\infty(M \times [0, T_{max}), N)$. Moreover, if $T_{max} < +\infty$, then*

$$\lim_{t \uparrow T_0} \|\nabla u(\cdot, t)\|_{C^0(M)} = +\infty. \tag{5.5}$$

Proof. It is similar to the presentation by Ding [41]. To simplify the proof, we assume that $M = \mathbb{R}^n$ and $g = g_0$ (the Euclidean metric). Let

$$G(x,t) = \frac{1}{(4\pi t)^{\frac{n}{2}}} \exp\left(-\frac{|x|^2}{4t}\right), \ t > 0$$

be the fundamental solution to the heat equation:

$$\begin{aligned} (\partial_t - \Delta)G(x,t) &= 0, \ x \in \mathbb{R}^n, \ t > 0 \\ \lim_{t\downarrow 0^+} G(x,t) &= \delta_0(x), \ x \in \mathbb{R}^n. \end{aligned}$$

For $0 < T \le \infty$, let $u : \mathbb{R}^n \times [0,T) \to \mathbb{R}^L$ be the solution of the Cauchy problem

$$\begin{aligned} \partial_t u - \Delta u &= f, \ (x,t) \in \mathbb{R}^n \times (0,T) \\ u &= u_0, \ x \in \mathbb{R}^n. \end{aligned}$$

Then we have the representation formula (cf. [47])

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} G(x-y,t)u_0(y)\,dy \\ &\quad + \int_0^t \int_{\mathbb{R}^n} G(x-y,t-s)f(y,s)\,dyds. \end{aligned} \tag{5.6}$$

We now recall some basic properties of (5.6) and linear parabolic equations (cf. [120, 128]).
(i) If $u_0 \in C(\mathbb{R}^n, \mathbb{R}^L)$ and $f \in L^\infty(\mathbb{R}^n \times [0,T], \mathbb{R}^L)$, then $u \in C(\mathbb{R}^n \times [0,T], \mathbb{R}^L)$ and $u(\cdot,t) \in C^{1,\alpha}(\mathbb{R}^n)$ uniformly in $t \in [\epsilon, T]$ for any $\epsilon > 0$.
(ii) For $\alpha \in (0,1)$, if $f(\cdot,t) \in C^\alpha(\mathbb{R}^n)$ uniformly in $t \in [\epsilon,T]$ for any $\epsilon > 0$, then $u_t, \nabla u, \nabla^2 u \in C(\mathbb{R}^n \times [\epsilon,T], \mathbb{R}^L)$.

Note that if $u \in C^\infty(\mathbb{R}^n \times [0,T), N)$ solves (5.3) and (5.4), then (5.6) implies

$$\begin{aligned} u(x,t) &= \int_{\mathbb{R}^n} G(x-y,t)u_0(y)\,dy \\ &+ \int_0^t \int_{\mathbb{R}^n} G(x-y,t-s)A(u)(\nabla u, \nabla u)(y,s)\,dyds. \end{aligned} \tag{5.7}$$

For $\epsilon > 0$, define

$$X_\epsilon = \left\{u \in C^0(\mathbb{R}^n \times [0,\epsilon], \mathbb{R}^L) \mid u(\cdot,0) = u_0, \ u(\cdot,t) \in C^1(\mathbb{R}^n), \ 0 \le t \le \epsilon\right\},$$

and endow X_ϵ with the norm

$$\|u\|_{X_\epsilon} := \|u\|_{C^0(\mathbb{R}^n \times [0,\epsilon])} + \sup_{t\in[0,\epsilon]} \|\nabla u(\cdot,t)\|_{C^0(\mathbb{R}^n)}, \ \forall u \in X_\epsilon.$$

Define $T : X_\epsilon \to X_\epsilon$ by

$$\begin{aligned} Tu(x,t) &= \int_{\mathbb{R}^n} G(x-y,t)u(y,0)\,dy \\ &+ \int_0^t \int_{\mathcal{R}^n} G(x-y,t-s)A(u)(\nabla u, \nabla u)(y,s)\,dyds, (x,t) \in \mathbb{R}^n \times [0,\epsilon). \end{aligned}$$

For $u \in X_\epsilon$, let

$$v_0(x,t) = \int_{\mathbb{R}^n} G(x-y,t)u(y,0)\,dy \quad \text{for } (x,t) \in \mathbb{R}^n \times [0,\epsilon).$$

For $\delta > 0$, let

$$B_\delta = \left\{u \in X_\epsilon \mid \|u - v_0\|_{X_\epsilon} \le \delta\right\}$$

be the ball in X_ϵ with center v_0 and radius δ.
We want to show that for any $\delta > 0$, there exists $\epsilon > 0$ such that
(i) $T : B_\delta \to B_\delta$.
(ii) There exists $\beta \in (0,1)$ such that

$$\|Tu - Tv\|_{X_\epsilon} \le \beta \|u - v\|_{X_\epsilon} \quad \text{for any } u, v \in B_\delta.$$

Proof. For $u \in B_\delta$, we have

$$\|u\|_{X_\epsilon} \le \|u - v_0\|_{X_\epsilon} + \|v_0\|_{X_\epsilon} \le C_0,$$

and

$$Tu - v_0 = \int_0^t \int_{\mathbb{R}^n} G(x-y,t-s)A(u)(\nabla u, \nabla u)(y,s)\,dyds.$$

Hence for any $(x,t) \in \mathbb{R}^n \times [0,\epsilon)$, we have

$$\begin{aligned} |Tu - v_0|(x,t) &\le C_1 \int_0^t \int_{\mathbb{R}^n} G(x-y,t-s)|\nabla u|^2(y,s)\,dyds \\ &\le C_1 \sup_{t\in[0,\epsilon]} \|\nabla u\|^2_{C^0(\mathbb{R}^n)} \int_0^\epsilon \int_{\mathbb{R}^n} G(x-y,t-s)\,dyds \\ &\le C_1\epsilon, \end{aligned} \tag{5.8}$$

and

$$\begin{aligned} |\nabla(Tu - v_0)|(x,t) &\le C_2 \int_0^t \int_{\mathbb{R}^n} |\nabla_x G|(x-y,t-s)|\nabla u|^2(y,s)\,dyds \\ &\le C_2 \sup_{t\in[0,\epsilon]} \|\nabla u\|^2_{C^0(\mathbb{R}^n)} \int_0^\epsilon \int_{\mathbb{R}^n} |\nabla_x G|(x-y,t-s)\,dyds \\ &\le C_2\epsilon. \end{aligned} \tag{5.9}$$

Therefore for any $\delta > 0$, there exists $\epsilon > 0$ such that $T : B_\delta \to B_\delta$.
For (ii), let $u, v \in B_\delta$, then we have

$$\begin{aligned} & |Tu - Tv|(x,t) \\ \le\ & \int_0^t \int_{\mathbb{R}^n} G(x-y,t-s)\,|A(u)(\nabla u,\nabla u) - A(v)(\nabla v,\nabla v)|\,(y,s) \\ \le\ & C \int_0^t \int_{\mathbb{R}^n} G(x-y,t-s)\left\{|\nabla u|^2|u-v| + (|\nabla u| + |\nabla v|)\,|\nabla(u-v)|\right\} \\ \le\ & C\epsilon\,\|u - v\|_{X_\epsilon}. \end{aligned} \tag{5.10}$$

Similarly, we can show

$$|\nabla (Tu - Tv)| (x,t) \leq C\epsilon \|u - v\|_{X_\epsilon}, \ \forall (x,t) \in \mathbb{R}^n \times [0,\epsilon].$$

Hence (ii) holds.
It follows from (i) and (ii) that there exists a unique $u \in B_\delta$ such that $u = Tu$, namely $u : \mathbb{R}^n \times [0,\epsilon) \to \mathbb{R}^L$ solves (5.3) and (5.4).
Finally, we need to show $u(\mathbb{R}^n \times [0,\epsilon)) \subset N$. For this, consider $\rho(u) = |\Pi_N(u) - u|^2$, where $\Pi_N : N_\delta \to N$ is the smooth nearest point projection map. Note that $\rho(u)(\cdot, 0) = 0$, and

$$\partial_t(\rho(u)) - \Delta\rho(u) = -|\nabla (\Pi_N(u) - u)|^2 \leq 0. \tag{5.11}$$

Hence by the maximum principle we have $\rho(u) \equiv 0$.

Since (5.3) is a quasi-linear parabolic system, it is well-known (cf. [128]) that if

$$\lim_{t \uparrow \epsilon} \|\nabla u\|_{L^\infty(\mathbb{R}^n)} < +\infty,$$

then u can be extended to be a smooth solution of (5.3)-(5.4) beyond ϵ. Therefore there exists a maximal time interval $T_{\max} \in (0, +\infty]$ such that $u \in C^\infty(M \times [0, T_{\max}), N)$ and $T_{\max}$ satisfies (5.5). The proof is complete. □

Remark 5.2.2 The characterization (5.5) of the first finite singular time $T_{\max}$ is not optimal in higher dimensions. By employing the unique continuation theorem of backward heat equation by Escauriaza-Seregin-Sverák [53] to rule out the existence of a quasi-harmonic S^n whose gradient is in $L^n(\mathbb{R}^n)$, Wang [207] has shown that for $n \geq 3$, the first finite singular time can be characterized by

$$\limsup_{t \uparrow T_{\max}} \|\nabla u(t)\|_{L^n(M)} = +\infty.$$

5.3 Existence of global smooth solutions under $K^N \leq 0$

When the sectional curvature K^N is nonpositive, Eells-Sampson [49] have proved the following theorem on the existence of global smooth solutions.

Theorem 5.3.1 *Suppose (M, g) is compact without boundary and the sectional curvature K^N of N is non-positive. Then for any $u_0 \in C^\infty(M, N)$ the Cauchy problem (5.3) and (5.4) admits a unique, smooth solution $u \in C^\infty(M \times \mathbb{R}_+, N)$ which, as $t \to \infty$ suitably, converges to a harmonic map $u_\infty \in C^\infty(M, N)$ in $C^2(M, N)$.*

In order to prove this theorem, we first need the energy inequality for (5.3).

Lemma 5.3.2 *For any $0 < T \leq \infty$, if $u \in C^\infty(M \times [0,T), N)$ solves (5.3), then*

$$E(u(t)) + \int_0^t \int_M |\partial_t u|^2 \, dv_g \, dt = E(u_0), \ \forall t \in [0, T). \tag{5.12}$$

Proof. Since $A(u)(\nabla u, \nabla u) \perp T_u N$, whenence multiplying (5.3) by $\partial_t u$ and integrating over M, we obtain

$$\int_M |\partial_t u|^2 \, dv_g + \frac{d}{dt}(E(u(t))) = 0.$$

Integrating over $[0,t]$ leads to (5.3.2). □

Next we need the Bochner identity for (5.3).

Lemma 5.3.3 *If $u \in C^\infty(M \times [0,T), N)$ solves (5.3) and(5.4), then $e(u)$ satisfies*

$$(\partial_t - \Delta_g)\, e(u) = -\,|\nabla du|^2 - Ric^M(\nabla u, \nabla u) + R^N(\nabla u, \nabla u, \nabla u, \nabla u). \tag{5.13}$$

In particular, if $K^N \leq 0$, then it holds

$$(\partial_t - \Delta_g)\, e(u) \leq Ce(u) \tag{5.14}$$

for some $C > 0$ depending only on the Ricci curvature of M.

Proof. See also Ding [41]. For $x_0 \in M$, let (x_α) and (u^i) be normal coordinates center at $x_0 \in M$ and $u(x_0) \in N$ respectively. Then we have

$$\begin{aligned}
(\partial_t - \Delta_g)\, e(u) &= \langle \partial_t u_\alpha, u_\alpha \rangle - |u_{\alpha\beta}|^2 - \langle u_\alpha, u_{\beta\alpha,\beta} \rangle \\
&= \langle \partial_t u_\alpha, u_\alpha \rangle - |u_{\alpha\beta}|^2 - \langle u_\alpha, u_{\beta\beta,\alpha} \rangle - R^M_{\alpha\beta}(u_\alpha, u_\beta) \\
&= -\,|u_{\alpha\beta}|^2 + \left\langle u_\alpha, (\partial_t u - \Delta_g u)_\alpha \right\rangle - R^M_{\alpha\beta}(u_\alpha, u_\beta)
\end{aligned}$$

where we have used the Ricci identity

$$u_{\beta\alpha,\beta} = u_{\beta\beta,\alpha} + R^M_{\alpha\beta} u_\beta.$$

On the other hand, by (5.3) we have

$$\begin{aligned}
\left\langle u_\alpha, (\partial_t u - \Delta_g u)_\alpha \right\rangle &= \langle u_\alpha, (A(u)(\nabla u, \nabla u))_\alpha \rangle \\
&= -\,\langle \Delta_g u, A(u)(\nabla u, \nabla u) \rangle \\
&= \langle A(u)(\nabla u, \nabla u), A(u)(\nabla u, \nabla u) \rangle \\
&= \langle A(u)(u_\alpha, u_\alpha), A(u)(u_\beta, u_\beta) \rangle
\end{aligned}$$

where we have used

$$\langle \partial_t u, A(u)(\nabla u, \nabla u) \rangle = 0 \;\text{ and }\; \langle u_\alpha, A(u)(\nabla u, \nabla u) \rangle = 0.$$

For $u_{\alpha\beta}$, similar to §1.6 we have

$$|u_{\alpha\beta}|^2 = |P(u)(u_{\alpha\beta})|^2 + |A(u)(u_\alpha, u_\beta)|^2 = |\nabla du|^2 + |A(u)(u_\alpha, u_\beta)|^2.$$

Putting all these identities together, we obtain

$$\begin{aligned}
(\partial_t - \Delta_g)\, e(u) &= -\,|\nabla du|^2 - \mathrm{Ric}^M(\nabla u, \nabla u) \\
&\quad + \left\{ \langle A(u)(u_\alpha, u_\alpha), A(u)(u_\beta, u_\beta) \rangle - |A(u)(u_\alpha, u_\beta)|^2 \right\}.
\end{aligned}$$

This and the Gauss-Kodazi equation imply (5.13) and (5.14). Hence the proof is complete. □

We now recall Moser's Harnack inequality for subsolutions of the heat equation (cf. [146, 120]). Let $\mathcal{L} = \partial_t - \Delta_g$ be the heat operator on M. For $z_0 = (x_0, t_0) \in M \times (0, +\infty)$ and $0 < R < \min\{i_M, \sqrt{t_0}\}$, denote by $P_R(z_0)$ the parabolic cylinder

$$P_R(z_0) = \left\{ z = (x,t) \in M \times (0,+\infty) \mid |x - x_0| \leq R,\ t_0 - R^2 \leq t \leq t_0 \right\}.$$

Then we have

Lemma 5.3.4 *Suppose $v \in C^\infty(P_R(z_0))$ is nonnegative and satisfies*

$$\mathcal{L}v \leq Cv \quad \text{in } P_R(z_0)$$

for some $C > 0$. Then there exists $C_1 > 0$ such that

$$v(z_0) \leq C_1 R^{-(n+2)} \int_{P_R(z_0)} v\, dz. \tag{5.15}$$

Proof of Theorem 5.3.1.

Applying Lemma 5.3.4 to (5.14), we have for any $z_0 = (x_0, t_0) \in M \times (0, +\infty)$ and $R > 0$ small,

$$\begin{aligned} e(u)(z_0) &\leq CR^{-(n+2)} \int_{P_R(z_0)} e(u)\, dz \\ &\leq CR^{-(n+2)} \int_{t_0 - R^2}^{t_0} E(u(t))\, dt \\ &\leq CR^{-n} E(u_0) \end{aligned} \tag{5.16}$$

Hence $|\nabla u|$ is uniformly bounded on $M \times [0, T]$. Hence, by the higher order regularity of linear parabolic equations (cf. [120]), we conclude that $u \in C^\infty(M \times [0, +\infty), N)$ and

$$\sup_{t \in [0,+\infty)} \|\nabla^m u\|_{C^0(M)} \leq C(m, M, N, u_0), \ \forall m \geq 1. \tag{5.17}$$

Since by (5.3.2)

$$\int_0^t \int_M |\partial_t u|^2 \leq E(u_0) < +\infty, \ \forall t > 0,$$

we have

$$\lim_{t \uparrow +\infty} \int_{t-2}^t \int_M |\partial_t u|^2 = 0.$$

Moreover, by (5.23) we have

$$(\partial_t - \Delta_g) |\partial_t u|^2 = -|\nabla \partial_t u|^2 + R^N(u)(\nabla u, \partial_t u, \nabla u, \partial_t u) \leq 0. \tag{5.18}$$

Hence, by Lemma 5.3.4 we have for any $0 < \alpha < 1$,

$$\|\partial_t u\|_{C^\alpha(M \times [t-1,t])} \leq C(\alpha) \|\partial_t u\|_{L^2(M \times [t-2,t])} (\to 0 \ \text{ as } t \to 0).$$

Hence we can choose $t_k \uparrow \infty$ be such that $u_t(\cdot, t_k) \to 0$ in $C^\alpha(M)$ and $u(\cdot, t_k) \to u_\infty$ in $C^2(M, N)$. This implies that $u_\infty \in C^2(M, N)$ satisfies

$$\Delta_g u_\infty + A(u_\infty)(\nabla u_\infty, \nabla u_\infty) = 0$$

so that u_∞ is a smooth harmonic map. □

Remark 5.3.5 When (M, g) has $\partial M \neq \emptyset$, one can consider the initial and boundary value problem for (5.3). More precisely, for a given map $\phi \in C^\infty(\overline{M}, N)$, consider $u : \overline{M} \times [0, +\infty) \to N$:

$$\begin{aligned} (\partial_t - \Delta_g)\, u &= A(u)(\nabla u, \nabla u) \quad \text{in } M \times (0, +\infty) && (5.19)\\ u|_{t=0} &= \phi \text{ on } M && (5.20)\\ u(x,t) &= \phi(x)\ x \in \partial M, t > 0. && (5.21) \end{aligned}$$

Then Hamilton [77] has extended [49] and showed that if $K^N \leq 0$, then there is a unique, smooth solution $u \in C^\infty(\overline{M} \times \mathbb{R}_+, N)$ to (5.19), (5.20, (5.21).

Remark 5.3.6 The harmonic map u_∞ constructed by Theorem 5.3.1 is minimizing energy in its homotopy class. This follows from the second variational formula for E in §1.6: Let $u_1, u_2 : M \to N$ be two smooth maps and let $\Phi : M \times [0,1] \to N$ be a geodesic homotopy between u_1 and u_2, i.e., a smooth homotopy such that $\Phi(\cdot, 0) = u_1$ and $\Phi(\cdot, 1) = u_2$ and, for any fixed $x \in M$, $\Phi(x, \cdot)$ is a geodesic, if u_1 is a harmonic map, then

$$\begin{aligned} & E(u_2) - E(u_1) \\ = & \int_0^1 d\sigma \int_0^\sigma ds \left\{ |\nabla_{\frac{d}{ds}} \nabla\Phi|^2 - R^N\left(\nabla\Phi, \frac{d\Phi}{ds}, \nabla\Phi, \frac{d\Phi}{ds}\right) \right\}. \end{aligned} \tag{5.22}$$

Hence if $K^N \leq 0$, then u_1 is an energy minimizer in its homotopy class.

The above remark plays an important role in Hartman's paper [86] in which it was proved that u_∞ is independent of the choices of $t_k \uparrow +\infty$.

Lemma 5.3.7 *Suppose that N has its sectional curvature $K^N \leq 0$ and $u \in C^\infty(M \times [0, +\infty), N)$ solves (5.3). Then $E(u(t))$ is a convex function of t.*

Proof. Direct calculations imply

$$\frac{d}{dt} E(u(t)) = -\int_M |\partial_t u|^2,$$

and

$$\partial_t |\partial_t u|^2 = \Delta_g |\partial_t u|^2 - |\nabla \partial_t u|^2 + R^N(u)(\nabla u, \partial_t u, \nabla u, \partial_t u). \tag{5.23}$$

Therefore by integration by parts we have

$$\begin{aligned} \frac{d^2}{dt^2} E(u(t)) &= -\frac{\partial}{\partial t} \int_M |\partial_t u|^2 = -\int_M \partial_t |\partial_t u|^2 \\ &= \int_M \left(|\nabla \partial_t u|^2 - R^N(u)(\nabla u, \partial_t u, \nabla u, \partial_t u) \right) \geq 0. \end{aligned}$$

This implies the convexity of $E(u(t))$. □

For $s_0 > 0$ and $0 < T \le +\infty$, let $\{\phi(x;s)\} \in C^\infty(M \times [0, s_0], N)$ and $u(x,t;s) \in C^\infty(M \times [0,T] \times [0, s_0], N)$ solve the family of (5.3):

$$\begin{aligned} (\partial_t - \Delta_g)\, u(x,t;s) &= A(u)\,(\nabla u, \nabla u)\,(x,t;s),\ x \in M,\ t > 0,\ s \in [0, s_0] \quad (5.24) \\ u(x,0;s) &= \phi(x,s),\ x \in M,\ s \in [0, s_0]. \quad (5.25) \end{aligned}$$

Then we have

Lemma 5.3.8 *Suppose that $K^N \le 0$. Then for any $s \in [0, s_0]$, $\sup_{x\in M} \left|\frac{\partial u}{\partial s}\right|^2 (x,t;s)$ is nonincreasing in t. Hence $\sup_{x\in M, s\in[0,s_0]} \left|\frac{\partial u}{\partial s}\right|^2 (x,t;s)$ is nonincreasing function of t.*

Proof. Direct calculations yield

$$(\Delta_g - \partial_t)\left|\frac{\partial u}{\partial s}\right|^2 = \left|\nabla \frac{\partial u}{\partial s}\right|^2 - R^N(u)\left(\nabla u, \frac{\partial u}{\partial s}, \nabla u, \frac{\partial u}{\partial s}\right) \ge 0.$$

Hence the conclusions follow from the maximum principle of the heat equation. □

Assume that $u_1, u_2 \in C^\infty(M.N)$ are homotopic, and $h : M \times [0,1] \to N$ is a smooth homotopy between $h(\cdot,0) = u_1$ and $h(\cdot,1) = u_2$.

Let $g(x,\cdot) : [0,1] \to N$ be the geodesic from $u_1(x)$ to $u_2(x)$ that is homotopic to $h(\cdot,x)$ and parameterized proportionally to acr length.

For $x \in M$, define $\hat{d}(u_1(x), u_2(x))$ to be the length of this geodesic arc $g(x,\cdot)$.

Corollary 5.3.9 *Under the same assumptions as Lemma 5.3.8, set $u(x,0;s) = g(x,s)$ for $x \in M$ and $s \in [0,1]$. Then*

$$\sup_{x\in M} \hat{d}\,(u(x,t;0), u(x,t;1))$$

is nonincreasing in t for all $t \in [0,T]$.

Proof. By construction, we have

$$\sup_{x\in M} \hat{d}^2\,(u(x,0;0), u(x,0;1)) = \sup_{x\in M} \hat{d}^2\,(g(x,0), g(x,1)) = \sup_{x\in M, s\in[0,1]} \left|\frac{\partial g}{\partial s}\right|^2.$$

On the other hand, for $t \in [0,T]$,

$$\hat{d}^2\,(u(x,t;0), u(x,t;1)) \le \sup_{s\in[0,1]} \left|\frac{\partial u}{\partial s}\right|^2 (x,t).$$

The claim then follows from Lemma 5.3.8. □

Now we can prove the uniqueness theorem by Hartman [86]

Theorem 5.3.10 *Assume that* $K^N \leq 0$ *and* $\phi \in C^\infty(M,N)$*, let* $u \in C^\infty(M \times [0,+\infty), N)$ *solve (5.3) with* $u(\cdot,0) = \phi$*. Then there exists a smooth harmonic map* $u_\infty \in C^\infty(M,N)$ *homotopic to* ϕ *such that* $\lim_{t\uparrow\infty} u(\cdot,t) = u_\infty$ *in* $C^2(M)$*.*

Proof. First by Theorem 5.3.1, there exist $t_i \uparrow +\infty$ and a smooth harmonic map $u_\infty \in C^\infty(M,N)$ homotopic to ϕ such that $u(t_i)$ converges to u_∞ in $C^2(M)$. Let $g(x,0) = u(x,t_i)$ and $g(x,1) = u_\infty(x)$ for $x \in M$. Then it is easy to see that $u(x,t_i+t)$ and $u_\infty(x)$ are smooth solutions of (5.3) with $g(x,0)$ and $g(x,1)$ as their initial data respectively.

By Corollary 5.3.9, we have

$$\hat{d}\left(u(x,t_i+t), u_\infty(x)\right) \leq \hat{d}\left(g(x,0), g(x,1)\right) = \hat{d}\left(u(x,t_i), u_\infty(x)\right),$$

hence for any $t \geq 0$, $u(t_i+t) \to u_\infty$ uniformly as $i \to \infty$. This implies that u_∞ is independent of any choice of $t_i \uparrow +\infty$. □

5.4 An extension of Eells-Sampson's theorem

In this section, we will present a generalization of Eells-Sampson's Theorem 5.3.1 by Ding-Lin [42], in which the nonpositivity assumption on K^N is replaced by a weaker assumption that the universal cover $\tilde{N}$ of N admits a strictly convex function of quadratic growth.

We begin with

Definition 5.4.1 A map $v \in C^\infty\left(\mathbb{R}^n \times (-\infty,0], N\right)$ is called an n-obstruction for (N,h), if (i) it is a heat flow of harmonic map on $\mathbb{R}^n \times \mathbb{R}_-$.
(ii) $|\nabla v|(0,0) > 0$ and

$$|\nabla v|(x,t) \leq |\nabla v|(0,0),\ \forall (x,t) \in \mathbb{R}^n \times \mathbb{R}_-. \tag{5.26}$$

(iii) there exists $E_0 > 0$ such that

$$R^{2-n}\int_{B_R} |\nabla v(t)|^2 \leq E_0,\ \forall R > 0 \text{ and } t \leq 0. \tag{5.27}$$

The following proposition illustrates that existence of an n-obstruction acts as an obstruction to the existence of global smooth solutions to (5.3).

Proposition 5.4.2 *Assume* (N,h) *does not admit any* n*-obstruction. Then (5.3) and (5.4) has a unique smooth solution* $u \in C^\infty(M \times \mathbb{R}_+, N)$*. Moreover, we have a priori estimate*

$$\|\nabla u(t)\|_{C^0(M)} \leq C\left(E(u_0)\right),\ \forall t \geq 1. \tag{5.28}$$

Proof. Suppose that it were false. Then u blows up at either $0 < T = T(u_0) < +\infty$ or $T = +\infty$. First, consider $0 < T < +\infty$. Then there exists $(x_i,t_i) \in M \times (0,T)$ such that $t_i \to T$ and

$$m_i = |\nabla u|(x_i,t_i) = \sup_{M\times[0,t_i]} |\nabla u|(x,t) \to +\infty.$$

Assume $x_i \to x_0 \in M$. Let $B_\rho(x_0)$ be the geodesic ball centered at x_0 with radius ρ. Denote $\lambda_i = m_i^{-1}$ and define

$$v_i(x,t) = u\left(x_0 + \lambda_i x, t_0 + \lambda_i^2 t\right) : B_{m_i\rho} \times \left(-m_i^2 t_0, 0\right] \to N.$$

Then v_i satisfies (i) $\partial_t v_i = \tau_i(v_i)$ on $B_{m_i\rho} \times (-m_i^2 t_0, 0] \to N$, where $\tau_i(v_i)$ is the tension field of v_i with respect to (N,h) and $(B_{m_i\rho}, g_i)$, where $g_i(x) = g(x_i + \lambda_i x)$ for $x \in B_{m_i\rho}$.
(ii) $1 = |\nabla v_i|(0,0) \geq |\nabla v_i|(x,t)$ for any $(x,t) \in B_{m_i\rho} \times (-m_i^2 t_0, 0]$. Since g_i converges to the Euclidean metric g_0 in $C^2_{\text{loc}}(\mathbb{R}^n)$, by the higher order regularity theory of parabolic equations we have

$$\|v_i\|_{C^k(B_R \times (-R^2,0])} \leq C\left(k,R\right), \ \forall R > 0, \ \forall i \geq i(R), \forall k \geq 1$$

for some constant $C(k,R) > 0$ depending on k and R. Hence, by a diagonal process, we can assume that $v_i \to v$ in $C^2_{\text{loc}}(\mathbb{R}^n \times (-\infty, 0], N)$. It is easy to see that v satisfies (i) and (ii).

To show v satisfies (5.27), for $t \in (-m_i^2 t_0, 0]$ and $R > 0$, denote $\bar{t}_i = t_i + \lambda_i(t + R^2)$, $z_i = (x_i, \bar{t}_i)$. We can compute

$$\begin{aligned}
R^{2-n} \int_{B_R} |\nabla v_i(t)|^2 &= (\lambda_i R)^{2-n} \int_{B_{\lambda_i R}(x_i) \times \{t_i + \lambda_i^2 t\}} |\nabla u(t)|^2 \\
&= (\lambda_i R)^{2-n} \int_{B_{\lambda_i R}(x_i)} |\nabla u|^2 \left(\bar{t}_i - (\lambda_i R)^2\right) \\
&\leq (4\pi)^{\frac{n}{2}} e^{\frac{1}{4}} \int_{B_{\lambda_i R}(x_i) \times \{\bar{t}_i - (\lambda_i R)^2\}} |\nabla u|^2 G_{z_i} \\
&\leq C\Phi\left(u, z_i, \lambda_i R\right),
\end{aligned}$$

where $\Phi(u, z_i, \lambda_i R)$ is given by (7.2). Set $\bar{r}^2 = \frac{1}{2}\min\left\{T, \rho^2\right\} > 0$. By the monotonicity inequality (7.4), we have

$$\begin{aligned}
\Phi\left(u, z_i, \lambda_i R\right) &\leq e^{c\bar{r}} \Phi\left(u, z_i, \bar{r}\right) + c\bar{r} E(u_0) \\
&\leq e^{c\bar{r}} \bar{r}^{2-n} \int_{M \times \{\bar{t}_i - \bar{r}^2\}} |\nabla u|^2 + c\bar{r} E(u_0) \\
&\leq \left(e^{c\bar{r}} \bar{r}^{2-n} + c\bar{r}\right) E(u_0).
\end{aligned}$$

This implies (5.27) and hence v is an n-obstruction. This contradicts the assumption and hence $T = +\infty$ and u is a global smooth solution.

Suppose that (5.28) were false. Then there exist a sequence of global smooth solutions $\{u_i\} \in C^\infty(M \times \mathbb{R}_+, N)$ to (5.3) with initial data $\{u_{i0}\} \subset C^\infty(M, N)$ such that $E(u_{i0}) \leq C_0$, but for a sequence $t_i \geq 1$ $\|\nabla u_i(t_i)\|_{C^0(M)} \to +\infty$. Then we can repeat the proceeding argument, with u replaced by u_i, to obtain the existence of an n-obstruction again. This completes the proof of Proposition 5.4.2. □

Now we prove the main theorem of this section (see [42]).

Theorem 5.4.3 *Let $(\tilde{N}, \tilde{h})$ be the universal covering of (N, h). Suppose that $\tilde{N}$ admits a strictly convex function $\rho \in C^2(\tilde{N})$ with quadratic growth, i.e., there are positive constants c_0, c_1, c_2 such that*

$$\nabla^2 \rho \geq c_0 \tilde{h} \quad on \quad \tilde{N}, \tag{5.29}$$

$$0 \leq \rho(y) \leq c_1 d^2_{\tilde{N}}(y, y_0) + c_2, \ \forall y \in \tilde{N} \tag{5.30}$$

for some $y_0 \in \tilde{N}$, where $d_{\tilde{N}}$ is the distance function on $(\tilde{N}, \tilde{h})$. Then there exists a unique, smooth solution $u \in C^\infty(M \times \mathbb{R}_+, N)$ which satisfies

$$\|\nabla u(t)\|_{C^0(M)} \leq C(E(u_0)), \ \forall t \geq 1, \tag{5.31}$$

and for suitable $t \to \infty$, $u(t)$ converges to a smooth harmonic map $u_\infty : M \to N$ in $C^2(M, N)$.

Proof. By Proposition 5.4.2, it suffices to show that (N, h) doesn't admit any n-obstruction.

Note that $\pi_1(N)$ acts isometrically on $\tilde{N}$ via the deck transformation, hence $\rho_\alpha = \rho \circ \alpha$, $\alpha \in \pi_1(N)$, also satisfies (5.29) and (5.30), with y_0 replaced by $\alpha^{-1}(y_0)$, on $\tilde{N}$. Denote by d the diameter of N. Then for any $\tilde{y} \in \tilde{N}$, there is $\alpha \in \pi_1(N)$ such that $d_{\tilde{N}}\left(\tilde{y}, \alpha^{-1}(y_0)\right) \leq d$.

Suppose that N supports an n-obstruction v. Then we can lift v into $\tilde{N}$ to obtain an obstruction $\tilde{v}$ for $(\tilde{N}, \tilde{h})$. Set $w = \rho_\alpha \circ \tilde{v} \in C^2\left(\mathbb{R}^n \times (-\infty, 0], \tilde{N}\right)$. Then by the chain rule (cf. [102]) we have

$$(\partial_t - \Delta) w = -\nabla^2 \rho_\alpha (\nabla \tilde{v}, \nabla \tilde{v}) \leq -c_0 |\nabla \tilde{v}|^2 = -c_0 |\nabla v|^2. \tag{5.32}$$

Note that by (5.29) and (5.30), $w(x,t) = O(|x|^2)$ as $x \to \infty$. Hence by the representation formula for the heat equation, we have for any $t_0 < 0$

$$\begin{aligned} (4\pi)^{\frac{n}{2}} w(0,0) \quad &\leq \quad |t_0|^{-\frac{n}{2}} \int_{\mathbb{R}^n} w(x, t_0) e^{-\frac{|x|^2}{4|t_0|}} \, dx \\ &- \quad c_0 \int_{t_0}^0 \int_{\mathbb{R}^n} |\nabla v|^2 \, |t|^{-\frac{n}{2}} e^{-\frac{|x|^2}{4|t|}} \, dx dt. \end{aligned}$$

Since $w \geq 0$, the implies

$$\int_{t_0}^0 \int_{\mathbb{R}^n} |\nabla v|^2 \, |t|^{-\frac{n}{2}} \, e^{-\frac{|x|^2}{4|t|}} \, dx dt \leq c_0^{-1} \, |t_0|^{-\frac{n}{2}} \int_{\mathbb{R}^n} w(x, t_0) e^{-\frac{|x|^2}{4|t_0|}} \, dx. \tag{5.33}$$

Now we claim that there exist $t_0 \in [-2, -1]$ and $\alpha \in \pi_1(N)$ such that

$$|t_0|^{-\frac{n}{2}} \int_{\mathbb{R}^n} w(x, t_0) e^{-\frac{|x|^2}{4|t_0|}} \, dx \leq c_3 \tag{5.34}$$

for some $c_3 > 0$ depending only on E_0, c_0, c_1, c_2. Assume for the moment that the claim is true. Then we have

$$\sum_{k=}^{\infty} \int_{4^{-k}}^{4^{-(k+1)}} \int_{\mathbb{R}^n} |\nabla v|^2 G_{(0,0)} \, dx dt \leq c_0^{-1} c_3 = c_4 < +\infty,$$

where $c_4 > 0$ only depends on E_0 and other given universal constants. Hence for $\epsilon_0 > 0$ given by proposition 7.1.4, there exists $k_0 \geq 1$ such that

$$I_{k_0} = \int_{4^{-k_0}}^{4^{-(k_0+1)}} \int_{\mathbb{R}^n} |\nabla v|^2 \, G_{(0,0)} \, dxdt \leq \epsilon_0^2.$$

Hence we have

$$|\nabla v|(0,0) \leq c_5 \tag{5.35}$$

for some $c_5 > 0$ depends only on E_0 and other universal constants.
On the other hand, it is easy to check that for any $\lambda > 0$, $v_\lambda(x,t) = v(\lambda x, \lambda^2 t) \in C^2(\mathbb{R}^n \times (-\infty, 0], N)$ is also an n-obstruction. Moreover, applying the above argument to v_λ, we obtain

$$|\nabla v_\lambda|(0,0) = \lambda |\nabla v|(0,0) \to +\infty,$$

which contradicts Definition 5.4.1(ii) .

Now we return to prove the claim. First note that by the ϵ_0-estimate for (5.3), we can find $t_0 \in [-2,-1]$, $x_0 \in B_1$, $\delta = \delta(E_0) > 0$ and $a = a(E_0) > 0$ such that

$$|\nabla v|(x, t_0) \leq a, \ \forall x \in B_\delta(x_0). \tag{5.36}$$

Also we can choose $\alpha \in \pi_1(N)$ such that

$$d_{\tilde{N}}\left(\tilde{v}(x_0, t_0), y_\alpha\right) \leq d, \quad y_\alpha = \alpha^{-1}(y_0).$$

Thus, by the triangle inequality, we have

$$\begin{aligned} d_{\tilde{N}}\left(\tilde{v}(x,t_0), y_\alpha\right) &\leq d_{\tilde{N}}\left(\tilde{v}(x,t_0), \tilde{v}(x_0,t_0)\right) + d_{\tilde{N}}\left(\tilde{v}(x_0,t_0), y_\alpha\right) \\ &\leq d_{\tilde{N}}\left(\tilde{v}(x,t_0), \tilde{v}(x_0,t_0)\right) + d. \end{aligned}$$

This and (5.36) imply that if $x = x_0 + r\theta$ for $r > 0$ and $\theta \in S^{n-1}$, then

$$d_{\tilde{N}}(\tilde{v}(x,t_0), y_\alpha) \leq \begin{cases} a\delta + d & \text{for } r \leq \delta \\ d_{\tilde{N}}(\tilde{v}(x_0 + r\theta, t_0), \tilde{v}(x_0 + \delta\theta, t_0)) + a\delta + d & \text{for } r > \delta \end{cases}$$

Since

$$w(x,t_0) \leq c_1 d_{\tilde{N}}^2\left(\tilde{v}(x,t_0), y_\alpha\right) + c_2,$$

we obtain that for some $c_6, c_7 > 0$

$$w(x,t_0) \leq \begin{cases} c_7 & \text{for } r \leq \delta \\ c_6 d_{\tilde{N}}^2(\tilde{v}(x_0 + r\theta, t_0), \tilde{v}(x_0 + \delta\theta, t_0)) + c_7 & \text{for } r > \delta \end{cases}$$

Therefore we have, for $t_0 \in [-2,-1]$,

$$
\begin{aligned}
&|t_0|^{-\frac{n}{2}} \int_{\mathbb{R}^n} w(x,t_0) e^{-\frac{|x|^2}{4|t_0|}} = |t_0|^{-\frac{n}{2}} \left\{ \int_{B_\delta(x_0)} + \int_{\mathbb{R}^n \setminus B_\delta(x_0)} \right\} w(x,t_0) e^{-\frac{|x|^2}{4|t_0|}} \\
&\le c_8 + c_9 \int_{\mathbb{R}^n \setminus B_\delta(x_0)} \left[\int_\delta^{|x-x_0|} \left| \frac{\partial v}{\partial r} \right| (x_0 + s\theta(x), t_0)\, ds \right]^2 e^{-\frac{|x|^2}{8}} \\
&\le c_8 + c_9 \int_\delta^\infty r^n e^{-\frac{r^2}{16}}\, dr \int_\delta^r \int_{S^{n-1}} \left| \frac{\partial v}{\partial r} \right|^2 (x_0 + s\theta, t_0)\, d\theta ds \\
&\le c_8 + c_{10} \delta^{1-n} \int_\delta^\infty r^n e^{-\frac{r^2}{16}} \left(\int_{B_\delta(x_0)} |\nabla v|^2 \right) dr \\
&\le c_8 + c_{11} E_0 \int_\delta^\infty r^n e^{-\frac{r^2}{16}} (r + |x_0|)^{n-2}\, dr \le c_{12}
\end{aligned}
$$

where we have used (5.27) in the last steps. This proves (5.34) and hence the proof is complete. □

Remark 5.4.4 It is well-known that if the sectional curvature of N is nonpositive and $\pi_1(N) = 0$, then $\rho \equiv d^2_{\tilde{N}}(\cdot, y_0)$ is strictly convex and the conditions (5.29) and (5.30) of Theorem 5.4.3 are satisfied. Consequently, Theorem 5.4.3 gives an alternative proof of Theorem 5.3.1.

As an application of Theorem 5.4.3, we can prove the existence of unstable harmonic maps into (N,h) admitting no n-obstructions.

Theorem 5.4.5 *Suppose that (N,h) does not admit any n-obstruction. Assume that there are two different harmonic maps u_1 and u_2 from (M,g) to (N,h) in a given homotopy class from M to N which are strictly stable in the sense that the second variations of E at u_1 and u_2 is positive. Then there exists a harmonic map of mountain pass type in the same homotopy class.*

Proof. Since u_i $(i = 1,2)$ are strictly stable harmonic maps, it follows from §1.6 that there exist neighborhoods U_i of u_i in $C^2(M,N)$, $i = 1,2$, such that

$$
E(u_i) < E(v), \ \forall v \in \overline{U_i}, \ v \neq u_i. \tag{5.37}
$$

Note that if $R > 0$ is sufficiently large so that $V_R \cap U_i \neq \emptyset$, where

$$
V_R \equiv \left\{ v \in C^3(M,N) \mid \|v\|_{C^3(M)} \le R \right\},
$$

then there exists $\delta_R > 0$ such that

$$
E(v) \ge E(u_i) + \delta_R, \ \forall v \in \partial U_i \cap V_R. \tag{5.38}
$$

In fact, suppose (5.38) were false. Then there exists $v_j \in V_R \cap \partial U_i$ such that $E(v_j) \to E(u_i)$ and $v_j \to v_0$ in $C^2(M,N)$. It follows that $v_0 \in \partial U_i$ and $E(v_0) = E(u_i)$, which contradicts (5.37).

Let $m = \max\{E(u_1), E(u_2)\}$ and

$$\Gamma = \left\{\gamma \in C([0,1], C^1(M,N)) \mid \gamma(0) = u_1,\ \gamma(1) = u_2\right\}.$$

Set

$$c = \inf_{\gamma\in\Gamma} \max_{s\in[0,1]} E(\gamma(s)).$$

Let $\gamma_j \in \Gamma$ be such that

$$c_j = E(\gamma_j(s(j))) = \max_{s\in[0,1]} E(\gamma_j(s)) \to c, \text{ as } j \to \infty,$$

where $s(j) \in [0,1]$. By Theorem 5.4.3, there exist global smooth solutions $u_{j,s}(t) \in C^\infty(M \times \mathbb{R}_+, N)$ to (5.3) with initial conditions $u_{j,s}(0) = \gamma_j(s)$. Then we have $u_{j,s}(t) \in \Gamma$ for any $t > 0$. Moreover, by the higher order regularity theory we have

$$\|u_{j,s}(t)\|_{C^3(M)} \le R = R(c) < +\infty,\ \forall t \ge 1,$$

where $R > 0$ depends on the value c.
Therefore, for $t \ge 1$, $u_{j,s}(t) \in C([0,1], C^1(M,N))$ is a continuous curve in V_R joining u_1 and u_2. In particular,

$$c_j = \max_{s\in[0,1]} E(\gamma_j(s)) \ge \max_{s\in[0,1]} E(u_{j,s}(t)) \ge m + \delta_R. \tag{5.39}$$

Taking $j \to \infty$, we get $c \ge m + \delta_R > m$. By a compactness argument, it is not hard to show that for any $j \ge 1$ $u_j(t) = u_{j,s(j)}(t)$ satisfies

$$c_j \ge E(u_j(t)) \ge c,\ \forall t > 0.$$

As $u_j(t)$ subconverges to a smooth harmonic map $v_j : M \to N$ in $C^2(M,N)$ as $t \to +\infty$, we have that $c_j \ge E(v_j) \ge c$. Also, by the estimate on harmonic maps, we may assume that v_j converges to a smooth harmonic map $v : M \to N$ in $C^2(M,N)$. Hence $E(v) = c > m$ and v is an unstable harmonic map. $\square$

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Chapter 6

Bubbling analysis in dimension two

Since the Dirichlet energy functional is conformally invariant in dimension two and the conformal group is not compact in dimension two in general, it is well known that the moduli space of harmonic maps in dimension two may be noncompact. Hence it is a very important question to understand the limiting behavior of sequences of solutions to harmonic maps and their evolutional problems in dimension two. Sacks-Uhlenbeck [164] have made a pioneering work in this direction. More precisely, they have developed a powerful blow up technique to the study of harmonic maps in dimension two and discovered that the failure of strong convergence of solutions of harmonic maps comes from the energy concentration at finitely many points. As a consequence, they have established the existence of branched minimal immersions of S^2 in Riemannian manifolds. Subsequentially, Struwe [193] has developed similar techniques in the context of heat flow of harmonic maps in dimensions two, and established the existence of solutions to the heat flow of harmonic maps with at most finitely many singularities. Moreover, the singularity occurs again due to possible energy concentration.

Bubbling analysis in dimension two is concerned with two main issues: (i) whether the total energy loss can be counted by the sum of energies of finitely many bubbles, i.e. nontrivial harmonic maps from S^2, (ii) whether attaching all possible bubbles to the weak limit gives uniform convergence. The first issue is refereed as energy identity and the second issue is refereed as bubble tree convergence.

For harmonic maps from surfaces, Jost [103] and Parker [152] have independently proved both energy identity and bubble tree convergence. Qing [157], Ding-Tian [43], Wang [206], Qing-Tian [158], and Lin-Wang [133] have studied both issues for either heat flows of harmonic maps or approximate harmonic maps with their tension fields bounded in L^2 in great lengths. However, when dealing with approximate harmonic maps in dimension two, it turns out that the L^2-space for the tension fields is not conformally invariant. Instead, L^1-space seems to be the nature space for the tension field. But Parker [152] has provided an example where the approximate harmonic maps have tension fields bounded in L^1 but the energy identity doesn't hold. This prompts us to search for the most suitable condition on the tension field such that both the energy identity and bubble tree convergence hold. In this aspect, Lin-Wang [135] have obtained an essentially optimal condition on the tension field.

This chapter is devoted to present some major ideas and results for bubbling analysis in dimension two. It is organized as follows. In §6.1, we outline Sacks-Uhlenbeck's work in [164]. In §6.2, we present Struwe's work in [193]. In §6.3, we present the example by Chang-Ding-Ye [23] on finite time singularity of heat flow of harmonic maps in dimension two. In §6.4, we present the proof of Lin-Wang [133] on bubble tree convergence of heat flow of harmonic maps at time infinity. In §6.5, we present the work by Lin-Wang [135] on approximate harmonic maps with tension fields bounded in L^p for $1 < p \leq 2$.

6.1 Minimal immersion of spheres

In a pioneering work, Sacks-Uhlenbeck [164] have introduced the blow-up analysis of harmonic maps in dimensions two and established the existence of branched minimal immersion of S^2 into any compact Riemannian manifold without boundary. The techniques developed by [164] have been profound in applications to many other geometric variational problems. The goal of this section is to introduce some basic ideas of [164].

Throughout this section, (M, g) is assumed to be a compact Riemannian surface with or without boundary and $N \subseteq \mathbb{R}^L$ is a compact Riemannian manifold without boundary. For any $\alpha \geq 1$, Sacks-Uhlenbeck [164] have introduced

$$E_\alpha(u) := \int_M \left(1 + |\nabla u|^2\right)^\alpha \, dv_g$$

over

$$W^{1,2\alpha}(M, N) := \left\{v : M \to N \mid \nabla v \in L^{2\alpha}\left(M, \mathbb{R}^{2L}\right)\right\}.$$

Note that if $\alpha = 1$, then critical points of $E_1(u) = \mathrm{Vol}(M) + E(u)$ are harmonic maps. E_α can be viewed as subcritical approximations of E_1 for $\alpha > 1$: since $\dim(M) = 2$, the Sobolev embedding theorem implies that $W^{1,2\alpha}(M, N) \subseteq C^{1-\frac{1}{\alpha}}(M, N)$ and hence $W^{1,2\alpha}(M, N)$ is a C^2 separable Banach manifold. Moreover, $E_\alpha : W^{1,2\alpha}(M, N) \to \mathbb{R}_+$ satisfies the Palais-Smale condition (C) (see Palais [151]). Hence, by the direct method in calculus of variations we have

Lemma 6.1.1 *For any $\alpha > 1$ and $\phi \in C^\infty(M, N)$, there exists $u_\alpha \in C^\infty(M, N)$ in the same homotopy class as ϕ (i.e. $u_\alpha \in [\phi]$) such that*

$$E_\alpha\left(u_\alpha\right) = \inf\left\{E_\alpha(v) \mid v \in W^{1,2\alpha}(M, N),\ [v] = [\phi]\right\}. \tag{6.1}$$

Moreover, u_α satisfies 2α-harmonic map equation:

$$div\left(\left(1 + |\nabla u_\alpha|^2\right)^{\alpha-1} \nabla u_\alpha\right) = \left(1 + |\nabla u_\alpha|^2\right)^{\alpha-1} A(u_\alpha)\left(\nabla u_\alpha, \nabla u_\alpha\right). \tag{6.2}$$

Proof. Set

$$C_\alpha = \inf\left\{E_\alpha(v) \mid v \in W^{1,2\alpha}(M, N),\ [v] = [\phi]\right\}.$$

Then

$$C_\alpha \le E_\alpha(\phi) \le \left(1 + \max_M |\nabla \phi|^2\right)^\alpha \mathrm{vol}(M).$$

Let $\{u_i\} \subset W^{1,2\alpha}(M,N) \cap [\phi]$ be a minimizing sequence of C_α. Then we can assume that $\int_M |\nabla u_i|^{2\alpha} \le 1 + C_\alpha$ for all $i \ge 1$. Since $u_i(M) \subset N$, u_i is uniformly bounded for all $i \ge 1$. Therefore, after taking possible subsequences, we may assume $u_i \to u_\alpha$ both in $C^0(M,N)$ and weakly in $W^{1,2\alpha}(M,N)$. This implies $[u_\alpha] = [\phi]$ and $E_\alpha(u_\alpha) \ge C_\alpha$. On the other hand, by the lower semicontinuity, we have

$$E_\alpha(u_\alpha) \le \liminf_{i\to\infty} E_\alpha(u_i) = C_\alpha.$$

By calculating the first variation of E_α, we can easily see that u_α is a weak solution of the Euler-Lagrange equation (6.2). By the Sobolev embedding theorem, $u_\alpha \in C^{1-\frac{1}{\alpha}}(M,N)$. Hence by [145] Theorem 1.11.1, $\nabla u_\alpha \in W^{1,2}(M,N)$ and hence $\nabla u_\alpha \in L^p(M)$ for any $1 \le p < +\infty$ by Sobolev embedding theorem again. Thus we can differentiate (6.2) and obtain

$$\Delta u_\alpha + (\alpha - 1)\frac{\langle \nabla^2 u_\alpha, \nabla u_\alpha\rangle \nabla u_\alpha}{1 + |\nabla u_\alpha|^2} + A(u_\alpha)\left(\nabla u_\alpha, \nabla u_\alpha\right) = 0. \tag{6.3}$$

Although this theorem is true for any $\alpha > 1$, we only give a simple proof for $\alpha - 1$ small. Write u_α as u. For $x \in M$, define

$$a^{kl}_{\beta\gamma}(x) = \delta_{\beta\gamma}\delta_{kl} + (\alpha - 1)\frac{u^k_\beta u^l_\gamma}{1 + |\nabla u|^2},\ \ 1 \le k,l \le L \text{ and } 1 \le \beta,\gamma \le 2.$$

Then (6.3) can be written as

$$\sum_{1\le\beta,\gamma\le 2, 1\le l\le L} a^{kl}_{\beta\gamma} u^l_{\beta\gamma} = -A(u)^k\left(\nabla u, \nabla u\right). \tag{6.4}$$

Note that

$$\left|a^{kl}_{\beta\gamma} - \delta_{\beta\gamma}\delta_{kl}\right| \le \alpha - 1,\ 1 \le \beta,\gamma \le 2,\ 1 \le k,l \le L.$$

Therefore, if $(\alpha - 1)$ is small, then the linear operator

$$L_\alpha(v) = \sum_{\beta\gamma}^{l} a^{kl}_{\beta\gamma} v^l_{\beta\gamma} : W^{2,4}(M,N) \to L^4(M,N)$$

has an inverse map. It follows $u \in W^{2,4}(M,N) \subset C^{1,\frac{1}{2}}(M,N)$. We can now treat the equation (6.4) as a linear equation in u with Hölder continuous coefficients. Hence by the Schauder theory (cf. [72]) we have $u \in C^{2,\delta}(M,N)$ for any $0 < \delta < 1$. The smoothness of u then follows from [145] Theorem 5.63. □

Since all the estimates are local, we assume $M = \Omega \subset \mathbb{R}^2$ when we discuss the analytic estimates. A refinemet of the above argument can yield

Lemma 6.1.2 *Let $D \subset \mathbb{R}^2$ be a unit ball and $u : D \to N$ be a critical point of E_α. If $(\alpha - 1)$ is sufficiently small depending on $1 < p < +\infty$, then for any smaller discs $D' \subset D$, we have*

$$\|\nabla u\|_{L^p(D')} \leq C(p, D', D, \|\nabla u\|_{L^4(D)}) \|\nabla u\|_{L^4(D)} \,. \tag{6.5}$$

Proof. Let $\phi \in C_0^\infty(D)$ be such that $\phi = 1$ on D', and choose a suitable coordinate system of $\mathbb{R}^L$ so that $\int_D u = 0$. Multiplying (6.3) by ϕ and putting terms from commuting differentiation with multiplication by ϕ on the right hand side gives

$$\left|\Delta(\phi u) + (\alpha - 1)\frac{\langle \nabla^2(\phi u), \nabla u\rangle \nabla u}{1 + |\nabla u|^2}\right| \leq |A(u)(\nabla(\phi u), \nabla u)| + k(\phi)\left(|u| + |\nabla u|\right),$$

where $k(\phi)$ depends on $\nabla^2\phi$, $\|A\|_{C^0}$ and $\|u\|_{L^\infty(M)}$. For all $1 < p < +\infty$, $W^{2,p}$-estimate (cf. [72]) implies

$$\begin{aligned} \|\Delta(\phi u)\|_{L^p(D)} &\leq (\alpha - 1)\|\phi u\|_{W^{2,p}(D)} + \|A\|_{C^0} \||\nabla(\phi u)||\nabla u|\|_{L^p(D)} \\ &+ k(\phi)\|u\|_{W^{1,p}(D)} \,. \end{aligned} \tag{6.6}$$

Let $c(p)$ be the operator norm of $\Delta^{-1} : L^p(D) \to W^{2,p} \cap W_0^{1,2}(D)$. Then we get

$$\begin{aligned} \left(c(p)^{-1} - (\alpha - 1)\right)\|\phi u\|_{W^{2,p}(D)} &\leq \|A\|_{C^0} \||\nabla(\phi u)||\nabla u|\|_{L^p(D)} \\ &+ k(\phi)\|u\|_{W^{1,p}(D)} \,. \end{aligned} \tag{6.7}$$

Now let $p = 2$. For $\alpha - 1 < c(2)^{-1}$ we get

$$\begin{aligned} (c(2)^{-1} - (\alpha - 1))\|\phi u\|_{W^{2,2}(D)} &\leq \|A\|_{C^0}\|\nabla(\phi u)\|_{L^4(D)}\|\nabla u\|_{L^4(D)} \\ &\quad +k(\phi)\|u\|_{W^{1,2}(D)}. \end{aligned}$$

This gives a bound on $\|u\|_{W^{2,2}(D'')}$, where $D'' = \{x \in D \mid \phi(x) = 1\}$. By the Sobolev embedding, this gives a bound on $\|u\|_{W^{1,p}(D'')}$ for any $1 < p < +\infty$. Repeat (6.7) for any p with ϕ now having support in D''. If $c(p)^{-1} > (\alpha - 1)$, then we get a bound on $\|u\|_{W^{2,p}}$ in the interior of D''. □

A further refinement of the above argument yields the following apriori gradient estimates, which plays a crucial role in the analysis.

Lemma 6.1.3 *Let $B \subseteq \mathbb{R}^2$ be a unit ball, there are $\epsilon_0 = \epsilon(N) > 0$ and $\alpha_0 > 1$ such that if $u \in C^\infty(B, N)$ is a critical point of E_α, $E_\alpha(u, B) \leq \epsilon_0^2$ and $1 \leq \alpha < \alpha_0$, then there is an estimate uniform in $1 \leq \alpha < \alpha_0$*

$$\|\nabla u\|_{W^{1,p}(B')} \leq C\left(p, B', B\right)\|\nabla u\|_{L^2(B)} \,, \quad \forall 1 < p < +\infty \tag{6.8}$$

for any smaller disk $B' \subset B$. In particular,

$$\sup_{B_{\frac{1}{2}}} |\nabla u| \leq C\|\nabla u\|_{L^2(B)} \,. \tag{6.9}$$

Proof. By the Sobolev embedding, (6.9) follows directly from (6.8). By Lemma 6.1.2, we need to control $\|\nabla u\|_{L^4(B'')}$ for any small disk $B'' \subset B$. Again assume $\int_B u = 0$ and apply (6.7) with $p = \frac{4}{3}$. By the Sobolev embedding $W^{2,\frac{4}{3}}(B) \subset W^{1,4}(B)$, we have

$$\begin{aligned}\left\||\nabla u||\nabla(\phi u)|\right\|_{L^{\frac{4}{3}}(B)} &\leq \|\nabla u\|_{L^2(B)} \|\nabla(\phi u)\|_{L^4(B)} \\ &\leq C\|\nabla u\|_{L^2(B)} \|\phi u\|_{W^{2,\frac{4}{3}}(B)}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\left(c(\frac{4}{3})^{-1} - (\alpha-1)\right) \|\phi u\|_{W^{2,\frac{4}{3}}(B)} &\leq C(\|A\|_{L^\infty}) \|\nabla u\|_{L^2(B)} \|\phi u\|_{W^{2,\frac{4}{3}}(B)} \\ &+ k(\phi)\|\nabla u\|_{L^2(B)} \qquad (6.10)\end{aligned}$$

If we choose $\epsilon_0 > 0$ such that

$$\epsilon_0 \leq \frac{1}{2}\frac{[(c(\frac{4}{3})^{-1} - (\alpha-1))]}{C(\|A\|_{L^\infty})^{-1}},$$

then we get an estimate on $\|\phi u\|_{W^{2,\frac{4}{3}}(B)}$ and $\|\phi u\|_{W^{1,4}(B)}$ as well. This finishes the proof. □

Another crucial ingredient in the analysis is the removability of isolated singularity for harmonic maps.

Theorem 6.1.4 *If $u \in C^\infty(B_1 \setminus \{0\}, N)$ is a harmonic map and $E_1(u) < +\infty$, then $u \in C^\infty(B_1, N)$.*

Before proving this theorem, we need a lemma.

Lemma 6.1.5 *Let $u : B_1 \setminus \{0\} \to N$ be a smooth harmonic map and $E_1(u) < +\infty$. Then for any $0 < r \leq 1$,*

$$\int_0^{2\pi} \left|\frac{\partial u}{\partial r}\right|^2 (r,\theta)\, d\theta = r^2 \int_0^{2\pi} \left|\frac{\partial u}{\partial \theta}\right|^2 d\theta. \qquad (6.11)$$

Proof. It suffices to prove (6.11) for $r = 1$. To do it, for $\epsilon > 0$ let $\phi(x) = \phi_\epsilon(|x|) \in C^\infty(B_1)$ be such that $\phi = 0$ in B_ϵ, $\phi = 1$ in $B \setminus B_{2\epsilon}$, $0 \leq \phi \leq 1$, and $|\nabla\phi| \leq \frac{2}{\epsilon}$. Multiplying (1.8) by $\phi(|x|)x \cdot \nabla u$ and integrating on B_1, we have by an integration by parts

$$\begin{aligned}0 &= \int_{B_1} \Delta u \cdot (\phi(|x|)x \cdot \nabla u) \\ &= \int_{B_1} \left((u_i \phi x_j u_j)_i - |x|\phi'(|x|)|u_r|^2 - \phi|\nabla u|^2 - \phi(|x|)x_j\left(|\nabla u|^2\right)_j\right) \\ &= \int_{\partial B_1} \phi(|x|)|u_r|^2 - \int_{B_1}\left(|x|\phi'(|x|)|u_r|^2 + \phi|\nabla u|^2\right) \\ &- \int_{\partial B_1} \frac{1}{2}\phi(|x|)|\nabla u|^2 + \int_{B_1}\left(\phi(|x|)|\nabla u|^2 + \frac{1}{2}|x|\phi'(|x|)|\nabla u|^2\right) \\ &= \int_{\partial B_1}\left(\phi(|x|)|u_r|^2 - \frac{1}{2}|\nabla u|^2\right) + \int_{B_1} |x|\phi'(|x|)\left(\frac{1}{2}|\nabla u|^2 - |u_r|^2\right).\end{aligned}$$

Since

$$\lim_{\epsilon\to 0}\left|\int_{B_1}|x|\phi'(|x|)\left(\frac{1}{2}|\nabla u|^2-|u_r|^2\right)\right|\le C\lim_{\epsilon\to 0}\int_{B_{2\epsilon}}|\nabla u|^2=0,$$

we have by taking $\epsilon\to 0$

$$\int_{\partial B_1}\left|\frac{\partial u}{\partial r}\right|^2=\frac{1}{2}\int_{\partial B_1}|\nabla u|^2,$$

which easily implies (6.11). □

Proof of Theorem 6.1.4:
By the conformal invariance of E_1, we can assume that $\int_{B_2}|\nabla u|^2\le\epsilon_0^2$, where ϵ_0 is given by Lemma 6.1.3. For any $0\neq x\in B_1$, since $B_{|x|}(x)\subset B_2$, $E(u,B_{|x|}(x))\le\epsilon_0^2$. Hence Lemma 6.1.3 implies that

$$|x|\,|\nabla u|\,(x)\le C\,\|\nabla u\|_{L^2(B_{|x|}(x))}\,(\le C\epsilon_0)\,. \tag{6.12}$$

We approximate u by a function $q=q(r)$ that depends only on the radial coordinate and is piecewise linear in $\log r$. For $m\ge 1$, let $q(2^{-m})=\frac{1}{2\pi}\int_0^{2\pi}u(2^{-m},\theta)\,d\theta$. Then q is harmonic for $r\in(2^{-m},2^{-m+1})$, $m\ge 1$. Now for $2^{-m}\le r\le 2^{-m+1}$, the maximum principle implies

$$\begin{aligned}|q(r)-u(r,\theta)| &\le 2\max\left\{|u(x)-u(y)|:2^{-m}\le|x|,\ |y|\le 2^{-m+1}\right\}\\ &\le 2^{-m+3}\max\left\{|\nabla u|(x):2^{-m}\le|x|\le 2^{-m+1}\right\}\\ &\le C\left(\int_{|x|\le 2^{-m+2}}|\nabla u|^2\right)^{\frac{1}{2}}\le C\epsilon_0.\end{aligned} \tag{6.13}$$

Multiplying the equation (1.8) of u by $(u-q)$ and integrating over B_1, we obtain

$$\begin{aligned}\int_{B_1}|\nabla(u-q)|^2 &= \sum_{m=1}^{\infty} r\int_0^{2\pi}(q(r)-u(r,\theta))\cdot\left(u_r(r,\theta)-q'(r)\right)|_{2^{-m}}^{2^{-m+1}}\\ &\quad+\int_{B_1}\Delta u\cdot(u-q)\,.\end{aligned} \tag{6.14}$$

The integral in θ of the boundary term containing $q'(r)$ vanishes because q is the average of u at 2^{-m}. Since u,q and u_r are all continuous, the terms with u_r cancel with succeeding and preceding terms and

$$\lim_{m\to\infty}\left(2^{-m}\int_0^{2\pi}\left(u(2^{-m},\theta)-q(2^{-m})\right)\cdot u_r(2^{-m},\theta)\,d\theta\right)=0.$$

We can also estimate

$$\left|\int_{B_1}\Delta u\cdot(u-q)\right|\le\|A\|_{L^\infty}\|\nabla u\|_{L^2(B_1)}^2\|u-q\|_{L^\infty(B_1)}\le C\epsilon_0\|\nabla u\|_{L^2(B_1)}^2.$$

Hence

$$\int_{B_1}|\nabla(u-q)|^2\le C\epsilon_0\int_{B_1}|\nabla u|^2+\left(\int_{r=1}|u-q|^2\,d\theta\right)^{\frac{1}{2}}\left(\int_{r=1}|u_r|^2\,d\theta\right)^{\frac{1}{2}}.$$

Since q does not depend on θ, it follows easily from Lemma 6.1.5 that

$$\frac{1}{2}\int_{B_1} |\nabla u|^2 = \int_{B_1} \frac{1}{r^2}|u_\theta|^2 \le \int_{B_1} |\nabla(u-q)|^2 .$$

By the Poincaré inequality, we have

$$\int_{r=1} |u-q|^2\, d\theta \le \int_{r=1} |u_\theta|^2\, d\theta = \frac{1}{2}\int_{r=1} |\nabla u|^2\, d\theta.$$

Therefore, if $\epsilon_0 > 0$ is such that $\delta_0 = 2C\epsilon_0 < 1$, then

$$(1-\delta_0)\int_{B_1} |\nabla u|^2 \le \int_{r=1} |\nabla u|^2\, d\theta.$$

If we translate this inequality into a disk of any radius, we get for $r \le 1$,

$$(1-\delta_0)\int_{B_r} |\nabla u|^2 \le r\int_{\partial B_r} |\nabla u|^2 = r\frac{d}{dr}\left(\int_{B_r} |\nabla u|^2\right).$$

Integrating this inequality gives $\int_{B_r} |\nabla u|^2 \le r^{1-\delta_0}\int_{B_1} |\nabla u|^2$. Applying Lemma 6.1.3 one last time, we get

$$|x|^2 |\nabla u|^2(x) \le C\int_{B_{2|x|}} |\nabla u|^2 \le C\,|x|^{1-\delta_0}\int_{B_1} |\nabla u|^2,\ \forall 0 < |x| < \frac{1}{2}.$$

This implies $\nabla u \in L^p(B_1)$ for some $p > 2$ and $u \in C^\alpha(B_1)$ for some $0 < \alpha < 1$. Hence by the higher order regularity theory, $u \in C^\infty(B, N)$. □

Combining these local estimates, one can prove

Theorem 6.1.6 *Let $u_\alpha \in C^\infty(M,N)$ be critical points of E_α and $E_\alpha(u_\alpha) \le B$ for $\alpha > 1$, and $u_\alpha \to u$ weakly in $W^{1,2}(M,N)$ as $\alpha \to 1$. Then there exist a subsequence $\{\beta\} \subset \{\alpha\}$ and finite number of points $\{x_1, \cdots, x_l\} \subset M$, where l depends on B, M and ϵ_0 such that $u_\beta \to u$ in $C^2_{\mathrm{loc}}(M \setminus \{x_1,\cdots,x_l\}, N)$. Moreover $u \in C^\infty(M,N)$ is a smooth harmonic map.*

Proof. For $\alpha > 1$, define Radon measures $\mu_\alpha = (1+|\nabla u_\alpha|^2)^\alpha\, dx$ on M. Then $\mu_\alpha(M) = E_\alpha(u_\alpha) \le B$ and we can assume that there is a nonnegative Radon measure μ on M such that $\mu_\alpha \to \mu$, $\alpha \to 1$, as convergence of Radon measures on M. By Fatou's lemma, $\mu = |\nabla u|^2\, dx + \nu$ for a nonnegative Radon measure ν on M. For $\epsilon_0 > 0$ given by Lemma 6.1.3, it is easy to see that there are a nonnegative integer $l \le \frac{B}{\epsilon_0^2}$ and a finite set $\Sigma = \{x_1,\cdots,x_l\} \subset M$ such that $\inf_{r>0}\mu(B_r(x)) \ge \epsilon_0^2$ for $x \in \Sigma$. For any $x_0 \in M \setminus \Sigma$, there is $r_0 > 0$ such that $\mu(B_{r_0}(x_0)) < \epsilon_0^2$. This implies that for α close to 1, $\mu_\alpha(B_{r_0}(x_0)) \le \epsilon_0^2$ and hence Lemma 6.1.3 implies

$$\|u_\alpha\|_{C^k(B_{\frac{r_0}{2}}(x_0))} \le C(k,\epsilon_0),\ \forall k \ge 1.$$

Hence there is a subsequence $\{\beta\} \subset \{\alpha\}$ such that $u_\beta \to u$ in $C^2\left(B_{\frac{r_0}{2}}(x_0), N\right)$. Since x_0 is arbitrary, we have $u_\beta \to u$ in $C^2_{\mathrm{loc}}(M \setminus \Sigma, N)$ and $u \in C^\infty(M\setminus\Sigma, N)$ is harmonic map. By Theorem 6.1.4, we have $u \in C^\infty(M,N)$. □

Definition 6.1.7 A smooth map $\omega : S^2 \to N$ is called a *bubble*, if it is a nontrivial harmonic map.

Now we want to show if $\Sigma \neq \emptyset$, then there exists at least one bubble. More precisely,

Theorem 6.1.8 *Let $u_\alpha : M \to N$ be a sequence of critical points of E_α such that $u_\alpha \to u$ in $C^2(M \setminus \{x_1, \cdots, x_l\}, N)$ but not in $C^2(M \setminus \{x_2, \cdots, x_l\}, N)$ for $\alpha \to 1$. Then there exists a bubble $\omega : S^2 \to N$ such that*

$$\omega(S^2) \subset \bigcap_{r>0} \left(\cap_{\alpha\to 1} \cup_{\beta\leq\alpha} u_\alpha\left(B_r(x_1)\right)\right).$$

Moreover

$$E(u) + E(\omega) \leq \overline{\lim}_{\alpha\to 1} E(u_\alpha). \tag{6.15}$$

Proof. Let $r_0 > 0$ be such that $B_{r_0}(x_1) \cap \{x_2, \cdots, x_l\} = \emptyset$. Let $x_\alpha \in B_{r_0}(x_1)$ be such that

$$b_\alpha \equiv |\nabla u_\alpha|(x_\alpha) = \max_{B_{r_0}(x_1)} |\nabla u_\alpha|(x).$$

Then $\lim_{\alpha\to 1} b_\alpha = \infty$. For, otherwise,

$$E_\alpha(u_\alpha, B_{r_0}(x_1)) \leq (1 + b_\alpha^2)^\alpha r_0^2 \leq \epsilon_0^2$$

so that Lemma 6.1.3 implies $u_\alpha \to u$ in $C^2(B_{\frac{r_0}{2}}(x_1), N)$, which contradicts the assumption. Define $v_\alpha(x) = u_\alpha(x_\alpha + b_\alpha^{-1}x) : B_{r_0 b_\alpha} \to N$. Then v_α is a critical point of

$$\hat{E}_\alpha(v) = \int \left(b_\alpha^{-2} + |\nabla v|^2\right)^\alpha,$$

and

$$|\nabla v_\alpha|(0) = 1 \text{ and } \max_{B_{r_0 b_\alpha}} |\nabla v_\alpha|(x) = 1.$$

Note that the disks $B_{r_0 b_\alpha}$ converges to $\mathbb{R}^2$ and the metrics on the disks converge to the Euclidean metric. By a diagonal process, we can assume that there is a harmonic map $v \in C^\infty(\mathbb{R}^2, N)$ such that $v_\alpha \to v$ in $C^2_{\rm loc}(\mathbb{R}^2, N)$. It follows that v is nontrivial, since $|\nabla v|(0) = 1$. Moreover, we have for any $0 < r \leq r_0$,

$$\begin{aligned} E(v, \mathbb{R}^2) + E\left(u, M \setminus B_r(x_1)\right) &\leq \overline{\lim}_{\alpha\to 1}\left\{E\left(v_\alpha, B_{r b_\alpha}\right) + E\left(u_\alpha, M \setminus B_r(x_1)\right)\right\} \\ &\leq \overline{\lim}_{\alpha\to 1} E(u_\alpha). \end{aligned}$$

By taking r to zero, this implies (6.15). Since $\mathbb{R}^2 = S^2 \setminus \{p\}$ conformally and $0 < E(v) < +\infty$, Theorem 6.1.4 implies that v extends to a nontrivial, smooth harmonic map $\omega : S^2 \to N$. □

As an application of these estimates, we present a proof of the following theorem independently by Lemaire [114], Schoen-Yau [177] and [164].

Theorem 6.1.9 *If $\dim(M) = 2$ and $\pi_2(N) = 0$, then any map $\phi \in C^\infty(M, N)$ is homotopic to a smooth harmonic map.*

Proof. Let $u_\alpha : M \to N$ be a minimizing map of E_α in the homotopy class $[\phi]$. By Theorem 6.1.6, there is a subsequence $\beta \to 1$ such that $u_\beta \to u$ in $C^2(M \setminus \{x_1, \cdots, x_l\}, N)$ with $u \in C^\infty(M, N)$ being a harmonic map. We claim $u_\beta \to u$ in $C^2(M, N)$. Suppose it were false. Then $l \geq 1$. For simplicity, assume $l = 1$. For a sufficiently small $\rho > 0$, we further assume that the metric g on $B_\rho(x_1)$ is the Euclidean metric. We perform a surgery of u_β in $B_\rho(x_1)$ as follows. Let $\eta \in C_0^\infty(B_\rho(x_1))$ be such that $0 \leq \eta \leq 1$, $\eta \equiv 1$ in $B_{\frac{\rho}{2}}(x_1)$, and $|\nabla \eta| \leq 8\rho^{-1}$. Define $v_\alpha : B_\rho(x_1) \to R^L$ by

$$v_\alpha(x) = \eta(x)u(x) + (1 - \eta(x))\, u_\alpha(x), \ \forall x \in B_\rho(x_1).$$

Then for any $\delta > 0$ there exists $\alpha_0 > 1$ such that for $1 < \alpha \leq \alpha_0$, we have

$$\max_{B_\rho(x_1)} \operatorname{dist}(v_\alpha(x), N) \leq \max_{B_\rho(x_1) \setminus B_{\frac{\rho}{2}}(x_1)} |u_\alpha(x) - u(x)| \leq \delta.$$

Therefore we can project v_α into N and get $w_\alpha(x) = \Pi_N(v_\alpha(x))$ for $x \in B_\rho(x_1)$. Now we define $\bar{u}_\alpha \in C^\infty(M, N)$ by letting

$$\bar{u}_\alpha(x) = \begin{cases} w_\alpha(x) \text{ for } x \in B_\rho(x_1) \\ u_\alpha(x) \text{ for } x \in M \setminus B_\rho(x_1). \end{cases}$$

Since $\pi_2(N) = \{0\}$, it is easy to see that u_α and $\bar{u}_\alpha$ are homotopic. Since u_α is an energy minimizer for E_α in its homotopy class, we have

$$E_\alpha\left(u_\alpha, B_\rho(x_1)\right) \leq E_\alpha\left(\bar{u}_\alpha, B_\rho(x_1)\right).$$

Note also that

$$\lim_{\alpha \to 1} E_\alpha\left(\bar{u}_\alpha, B_\rho(x_1)\right) = E\left(u, B_\rho(x_1)\right) \leq \pi\rho^2 \left\|\nabla u\right\|^2_{L^\infty(M)}.$$

Therefore we have

$$E_\alpha\left(u_\alpha, B_\rho(x_1)\right) \leq \pi\rho^2 \left\|\nabla u\right\|^2_{L^\infty(M)} \leq \frac{\epsilon_0^2}{2},$$

provided that α is sufficiently close to 1. Hence Lemma 6.1.3 implies that $u_\alpha \to u$ in $C^2(B_{\frac{\rho}{2}}(x_1), N)$, and we get a contradiction. This completes the proof □

To conclude this section, we present a proof of another main theorem by [164] on the existence of minimal spheres.

Theorem 6.1.10 *If the universal covering space of N is not contractible, then there exists a nontrivial harmonic map $\omega : S^2 \to N$.*

Proof. For $\alpha \downarrow 1$, since the universal covering space $\tilde{N}$ of N is not contractible, we can show that there are critical points $u_\alpha : S^2 \to N$ of E_α with $\epsilon \leq E_\alpha(u_\alpha) \leq C$ for some $\epsilon > 0$. Then we may assume, up to a subsequence, that there are p points $\{x_1, \cdots, x_p\} \subset S^2$ such that $u_\alpha \to u$ in $C^1(S^2 \setminus \{x_1, \cdots, x_p\}, N)$. By theorem 6.1.4 we have that $u \in C^\infty(S^2, N)$ is a smooth harmonic map. If $u \neq$constant, then we are done. Hence we may assume that $u =$ constant. Then we would have $p \geq 1$. Then theorem 6.1.8 implies that there exists a non constant harmonic map $\tilde{u} \in C^\infty(S^2, N)$ with $\tilde{u}(S^2) \cap_\alpha \overline{\cup_{\beta < \alpha} u_\alpha(S^2)}$. Hence we again have a nontrivial harmonic map from S^2. Any such harmonic map is a conformal, branched minimal immersion. □

6.2 Almost smooth heat flows in dimension two

In this section, we present the classical theorem by Struwe [193] on the existence of a global, weak solution of the heat flow of harmonic maps from a Riemannian surface into any compact Riemannian manifold, that is smooth away from finitely many possible singular points. Our presentation follows closely from that of Struwe [196].

More precisely, Struwe [193] has proved the following theorem.

Theorem 6.2.1 *Suppose M is a compact Riemannian surface without boundary, $N \subset \mathbb{R}^L$ is a compact Riemannian manifold without boundary. Then for $u_0 \in W^{1,2}(M, N)$ there exists a global weak solution $u : M \times [0, +\infty) \to N$ of (5.3)-(5.4) satisfying an energy inequality:*

$$E(u(t)) \leq E(u_0) \quad \textit{for all } t > 0. \tag{6.16}$$

Moreover, there exist an integer $K \geq 0$ depending only on M, N and $E(u_0)$ such that $u \in C^\infty(M \times (0, +\infty) \setminus \{(x_k, t_k)\}_{k=1}^K, N)$ for some $\{(x_k, t_k)\} \subset M \times (0, +\infty)$. The solution is unique in this class. At each singular point (x_j, t_j) for $1 \leq j \leq K$, there exists a bubble $\omega_j : S^2 \to N$, $x_j^k \to x_j$, $t_j^k \uparrow t_j$, $r_j^k \downarrow 0$ such that

$$u_k^j(x) = u\left(x_j^k + r_j^k x, t_j^k\right) : B_{(r_j^k)^{-1}}(\subset \mathbb{R}^2) \to N$$

converges to ω in $W^{2,2}_{\text{loc}}(\mathbb{R}^2, N)$. Finally, there exists $t_k \uparrow \infty$ such that $u(\cdot, t_k)$ converges weakly in $W^{1,2}(M, N)$ to a smooth harmonic map $u_\infty : M \to N$. The convergence is strong away from finitely many points $\{x_p^\infty\}_{p=1}^I$, where again harmonic spheres separate in the above sense. Moreover, $K + I \leq \frac{E(u_0)}{\epsilon_0^2}$, where

$$\epsilon_0^2 = \inf\left\{E(\omega) : \omega : S^2 \to N \quad \textit{is a bubble}\right\} > 0 \tag{6.17}$$

depends only on N.

The proof of Theorem 6.2.1 is divided into several lemmas. First, let's establish an energy quantization for bubbles.

Lemma 6.2.2 *Let ϵ_0 be given by (6.17). Then $\epsilon_0 > 0$ is a constant depending only on N.*

Proof. Suppose $\epsilon_0 = 0$. Then there exist a sequence of bubbles $\omega_i : S^2 \to N$, $i \geq 1$, such that

$$\int_{S^2} |\nabla \omega_i|^2 = o(1).$$

By the conformal invariance of E, we may assume

$$|\nabla \omega_i|((0, 0, 1)) = 1 \text{ for all } i \geq 1.$$

On the other hand, by Lemma 6.1.3 we have

$$|\nabla \omega_i|(z) \leq o(1), \;\; \forall z \in S^2.$$

We get a contradiction. □

Lemma 6.2.3 *For dim$(M) = 2$, $C^\infty(M,N)$ is dense in $W^{1,2}(M,N)$.*

Proof. By the partition of unit, it suffices to show that for $M = B_2(0) \subset \mathbb{R}^2$ and $u \in W^{1,2}(B_2(0), N)$, there is a sequence $\{u_i\} \subset C^\infty(B_1(0), N)$ such that $\|u_i - u\|_{W^{1,2}(B_1(0))} \to 0$. For any $\epsilon > 0$, let $u_\epsilon = u * \eta_\epsilon$ be a standard ϵ-mollification of u. By a modified Poincaré inequality (see [172]), we have for any $x \in B_1(0)$,

$$\epsilon^{-2} \int_{B_\epsilon(x)} |u(y) - u_\epsilon(x)| \, dy \le C \int_{B_\epsilon(x)} |\nabla u|^2(y) \, dy \le C\delta \, (<< 1) \,, \tag{6.18}$$

provided that $\epsilon = \epsilon(\delta) > 0$ is chosen to be sufficiently small. Hence $u_\epsilon(B_1(0)) \subset N_\delta$ and $u_i = \Pi_N(u_\epsilon) \in C^\infty(B_1(0), N)$. Moreover

$$\|\nabla(u_i - u)\|_{L^2(B_1(0))} \le \|\nabla u - \nabla u_\epsilon\|_{L^2(B_1(0))} + \|(\nabla \Pi_N(u_\epsilon) - 1)\, \nabla u_\epsilon\|_{L^2(B_1(0))} \to 0.$$

This finishes the proof. □

Another crucial ingredient is the following interpolation inequality (see, [127]).

Lemma 6.2.4 *For any $v \in W^{1,2}(\mathbb{R}^2)$, we have $v \in L^4(\mathbb{R}^2)$ and*

$$\|v\|^4_{L^4(\mathbb{R}^2)} \le C \, \|v\|^2_{L^2(\mathbb{R}^2)} \, \|\nabla v\|^2_{L^2(\mathbb{R}^2)} \, . \tag{6.19}$$

Proof. Note that $v \in W^{1,2}(\mathbb{R}^2)$ implies $|v|^2 \in W^{1,1}(\mathbb{R}^2)$. Hence, by the Sobolev embedding theorem we have

$$\left\| |v|^2 \right\|_{L^2(\mathbb{R}^2)} \le C \left\| \nabla |v|^2 \right\|_{L^1(\mathbb{R}^2)} .$$

This and the Hölder inequality imply (6.19). □

We now need a local energy inequality.

Lemma 6.2.5 *Let $u \in C^\infty(M \times [0,T], N)$ solve (5.3) and (5.4). For $x_0 \in M$ and $t_0 > 0$, let $R_0 < \frac{1}{2} \min\{i_M, \sqrt{t_0}\}$. Then for $R \le R_0$, it holds*

$$E\left(u(T), B_R(x_0)\right) \le E\left(u_0, B_{2R}(x_0)\right) + C\frac{T}{R^2} E(u_0) \tag{6.20}$$

for some $C = C(M,N) > 0$.

Proof. Let $\phi \in C_0^\infty(B_{2R}(x_0))$ be such that $0 \le \phi \le 1$, $\phi = 1$ on $B_R(x_0)$, and $|\nabla \phi| \le \frac{2}{R}$. Multiplying (5.3) by $\partial_t u \phi^2$ and integrating over M lead to

$$\begin{aligned} \int_M |\partial_t u|^2 \, \phi^2 + \frac{d}{dt}\left(\int_M e(u)\phi^2\right) &\le C \int_M |\nabla u| \, |\partial_t u| \, |\nabla \phi| \, |\phi| \\ &\le \frac{1}{2} \int_M |\partial_t u|^2 \, \phi^2 + C \int_M |\nabla u|^2 \, |\nabla \phi|^2 . \end{aligned}$$

Hence we have

$$\frac{d}{dt}\left(\int_M e(u)\phi^2\right) \le CR^{-2} E\left(u(t)\right) \le CR^{-2} E(u_0).$$

Integrating from 0 to T leads to (6.20). □

An immediate consequence of Lemma 6.20 is the following: for any $\epsilon_1 > 0$ and $u_0 \in C^\infty(M,N)$, there exists $T_1 > 0$ depending only on M, N and a maximal number $R_1 > 0$ for which

$$\sup_{x\in M} E\left(u_0, B_{2R_1}(x)\right) < \epsilon_1^2 \tag{6.21}$$

such that any smooth solution u of (5.3) and (5.4) satisfies

$$\sup_{x_0\in M, 0\le t\le T_1} E\left(u(t), B_{R_1}(x)\right) < 2\epsilon_1^2. \tag{6.22}$$

Indeed, $T_1 = \frac{\epsilon_1 R_1^2}{CE(u_0)}$ does it.
Let $R_1 > 0$ and $T_1 > 0$ be such that (6.21) and (6.22) hold. Then let $\{\phi_i\} \subset C_0^\infty(M)$ be a partition of unit associated with a finite cover of M by $B_{2R_1}(x_i)$ with finite overlap, $0 \le \phi_i \le 1$, $|\nabla\phi_i| \le \frac{2}{R_i}$, and $\sum_i \phi_i^4 = 1$.

Then, applying (6.19) we have

$$\begin{aligned}\int_M |\nabla u|^4 &= \sum_i \int_M |\nabla u|^4 \phi_i^4 \\ &\le C \sup_i E\left(u(t), B_{2R_1}(x_i)\right)\left(\int_M |\nabla^2 u|^2 + R_1^{-2}E(u_0)\right) \\ &\le C\epsilon_1 \left(\int_M |\nabla^2 u|^2 + R_1^{-2}E(u_0)\right).\end{aligned} \tag{6.23}$$

On the other hand, multiplying (5.3) by $\Delta_g u$ and integrating by parts lead

$$\int_0^t \int_M |\Delta_g u|^2 + E\left(u(t)\right) \le E(u_0) + C\int_0^t \int_M |\nabla u|^4. \tag{6.24}$$

Note that

$$\int_0^t \int_M \left|\nabla^2 u\right|^2 \le C\left(\int_0^t \int_M |\Delta_g u|^2 + tE(u_0)\right). \tag{6.25}$$

Combining (6.23) with (6.24) and (6.25), we have

$$\int_0^{T_1} \int_M \left|\nabla^2 u\right|^2 \le C\epsilon_1 \int_0^{T_1} \int_M \left|\nabla^2 u\right|^2 + C\left(1 + T_1 R_1^{-2}\right) E(u_0). \tag{6.26}$$

Thus for sufficiently small $\epsilon_1 > 0$, we have

$$\|u\|^2_{V^{T_1}(M)} \le C\left(1 + \frac{T_1}{R_1^2}\right) E(u_0) \tag{6.27}$$

where

$$\|u\|^2_{V^{T_1}(M)} := \sup_{0\le t\le T_1} E\left(u(t)\right) + \int_0^{T_1} \int_M \left(|\nabla^2 u|^2 + |u_t|^2\right). \tag{6.28}$$

Proof of Theorem 6.1.6: We outline the proof by dividing it into six steps.
Step 1. Local existence: For $u_0 \in W^{1,2}(M,N)$, let $\{u_{0m}\} \subset C^\infty(M,N)$ converge to u_0 strongly in $W^{1,2}(M,N)$. Let u_m solve (5.3) with the initial data u_{0m}. Then for $T_1 > 0$ such that (6.22) holds, we have

$$\|u_m\|^2_{V^{T_1}(M)} \le C\left(1 + \frac{T_1}{R_1^2}\right) E(u_0). \tag{6.29}$$

Therefore, we can assume that $u_m \to u$ weakly in $V^{T_1}(M)$. It is easy to check that u solves (5.3) with the initial data u_0.

Step 2. Uniqueness: Let $u, v \in V^T(M)$ solve (5.3) such that $u(0) = v(0) = u_0$. Then $w = u - v$ satisfies

$$|w_t - \Delta_g w| \le C\,|w|\left(|\nabla u|^2 + |\nabla v|^2\right) + C\,|\nabla w|\left(|\nabla u| + |\nabla v|\right). \tag{6.30}$$

Multiplying (6.30) by w and integrating over M leads

$$\begin{aligned}
& \frac{1}{2}\int_M |w|^2 + \int_0^t\int_M |\nabla w|^2 \\
\le\ & C\int_0^t\int_M |w|^2\left(|\nabla u|^2 + |\nabla v|^2\right) + C\int_0^t\int_M |w|\,|\nabla w|\left(|\nabla u| + |\nabla v|\right) \\
\le\ & C\,\|w\|^2_{L^4(M\times[0,t])}\left(\|\nabla u\|^2_{L^4(M\times[0,t])} + \|\nabla v\|^2_{L^4(M\times[0,t])}\right) \\
+\ & C\,\|w\|_{L^4(M\times[0,t])}\,\|\nabla w\|_{L^2(M\times[0,t])}\left(\|\nabla u\|_{L^4(M\times[0,t])} + \|\nabla v\|_{L^4(M\times[0,t])}\right) \\
\le\ & C\epsilon(t)\left\{\left(\int_0^t\int_M |w|^4\right)^{\frac{1}{2}} + \int_0^t\int_M |\nabla w|^2\right\} \\
\le\ & C\epsilon(t)\left\{\sup_{[0,t]}\int_M |w|^2 + \int_0^t\int_M |\nabla w|^2\right\}
\end{aligned}$$

where $\epsilon(t) \to 0$ as $t \to 0$. Hence the uniqueness follows.

Step 3. Global extension: It is clear that the first singular time $T = T_1$ can be characterized by

$$\overline{\lim_{t\uparrow T}}\ \sup_{x\in M} E\left(u(t), B_R(x)\right) \ge \epsilon_1,\ \forall R > 0. \tag{6.31}$$

Since $\partial_t u \in L^2(M \times [0,T])$ and $E(u(t)) \le E(u_0)$ for $0 < t < T$, we can show that there exists $u(\cdot,T) \in W^{1,2}(M,N)$ such that $u(\cdot,t) \to u(\cdot,T)$ weakly in $W^{1,2}(M,N)$ as $t \uparrow T$. In particular,

$$E(u(T)) \le \liminf_{s\uparrow T} E(u(s)) \le E(u(t)) \ \text{ for all } 0 \le t < T.$$

Now let $v : M \times [T, T+T_2) \to N$ be the local solution of (5.3) constructed by Step 1 with $v(x,T) = u(x,T)$, and define

$$w(t) = \begin{cases} u(t) & 0 \le t \le T \\ v(t) & T \le t \le T + T_2. \end{cases}$$

Then one can verify that $w : M \times [0, T + T_2) \to N$ is a weak solution of (5.3) which satisfies (6.16). By iteration we obtain a weak solution u on a maximal time interval $[0, \bar{T})$. If $\bar{T} < +\infty$, then the above argument allows us to extend u beyond $\bar{T}$. Hence $\bar{T} = \infty$.

Step 4. Finitely many singular points: To see this, let $T_0 > 0$ be the first singular time and define $\mathcal{S}(u, T_0)$ by

$$\mathcal{S}(u, T_0) = \bigcap_{R>0} \left\{ x \in M \mid \overline{\lim_{t \uparrow T_0}} E\left(u(t), B_R(x)\right) \geq \epsilon_1 \right\}.$$

Let $\{x_j\}_{j=1}^K$ be any finite subset of $\mathcal{S}(u, T_0)$. Then we have

$$\overline{\lim_{t \uparrow T_0}} \int_{B_R(x_j)} |\nabla u|^2 (x, t) \geq \epsilon_1, \ \forall R > 0, \ 1 \leq j \leq K.$$

Therefore, if we choose $R > 0$ such that $B_{2R}(x_j), 1 \leq j \leq K$, are mutually disjoint. Then by (6.20) we have

$$\begin{aligned} K\epsilon_1 &\leq \sum_{1 \leq i \leq K} \overline{\lim_{t \uparrow T_0}} E\left(u(t), B_R(x_j)\right) \\ &\leq \sum_{1 \leq i \leq K} \left(E(u(\tau), B_{2R}(x_j)) + \frac{\epsilon_1}{2} \right) \\ &\leq E(u(\tau)) + \frac{K\epsilon_1}{2} \end{aligned}$$

for any $\tau \in [T_0 - \frac{\epsilon_1 R^2}{2CE(u_0)}, T_0]$. Therefore we have $K \leq \frac{2E(u_0)}{\epsilon_1}$. This implies the finiteness of $\mathcal{S}(u, T_0)$. Moreover, we have

$$\begin{aligned} E\left(u(T_0)\right) &= \lim_{R \to 0} E\left(u(T_0), M \setminus \bigcup_{1 \leq j \leq K} B_{2R}(x_j) \right) \\ &\leq \lim_{R \downarrow 0} \overline{\lim_{t \uparrow T_0}} E\left(u(t), M \setminus \bigcup_{1 \leq j \leq K} B_{2R}(x_j) \right) \\ &\leq E(u(t)) - \lim_{R \downarrow 0} \sum_{1 \leq j \leq K} \liminf_{t \uparrow T_0} E\left(u(t), B_{2R}(x_j)\right) \\ &\leq E(u_0) - \sum_{1 \leq j \leq K} \lim_{R \downarrow 0} \overline{\lim_{t \uparrow T_0}} E\left(u(t), B_R(x_j)\right) \\ &\leq E(u_0) - K\epsilon_1. \end{aligned} \tag{6.32}$$

Now suppose $T_0 < \cdots < T_j$ are j singular times and let $K_0, \cdots, K_j$ be the number of singular points at each singular time respectively. Let $u_i = \lim_{t \uparrow T_i} u(t)$ for $0 \leq i \leq j$. Then by (6.32) we have

$$E(u_j) \leq E(u_{j-1}) - K_{j-1}\epsilon_1 \leq E(u_0) - \left(\sum_{1 \leq i \leq j} K_i \right) \epsilon_1.$$

This implies $\sum_{1\leq i\leq j} K_i \leq \frac{E(u_0}{\epsilon_1}$. Hence there are at most finitely many singular points.

Step 5. Smoothness away from finitely many points. Let $T_0 > 0$ be the first singular time and $x_0 \in M \setminus \mathcal{S}(u, T_0)$. Then there exists $r_0 > 0$ such that

$$\overline{\lim_{t\uparrow T_0}} E\left(u(t), B_{2r_0}(x_0)\right) \leq \epsilon_1.$$

Hence there exists $0 < r_1 \leq r_0$ such that

$$\sup_{[T_0 - r_1^2, T_0)} E\left(u(t), B_{r_1}(x_0)\right) \leq \epsilon_1. \tag{6.33}$$

Now we need

Proposition 6.2.6 *Suppose $u \in C^\infty(P_{r_0}(z_0), N)$ solves (5.3). Then there exist ϵ_0 and $C_0 > 0$ depending only on n, N such that if*

$$\sup_{[t_0 - r_0^2, t_0)} E\left(u(t), B_{r_0}(x_0)\right) \leq \epsilon_0,$$

then

$$\sup_{z \in P_{\frac{r_0}{2}}(z_0)} \left|\nabla u\right|(z) \leq C_0. \tag{6.34}$$

Proof. The ideas are similar to [166]. For simplicity, assume $z_0 = (0, 1)$ and $r_0 = 1$. Denote $P_r(z_0)$ by P_r. Let $\rho \in (0, 1)$ be such that

$$(1-\rho)^2 \sup_{P_\rho} e(u) = \max_{\sigma\in[0,1]} \left\{ (1-\sigma)^2 \sup_{P_\sigma} e(u) \right\}$$

and let $z_0 \in P_\rho$ satisfy

$$e(u)(z_0) = \sup_{P_\rho} e(u) = e_0.$$

Then either $e_0(1-\rho)^2 \leq 2$ so that

$$(\frac{1}{2})^2 \sup_{P_{\frac{1}{2}}} e(u) \leq e_0 \left(1-\rho\right)^2 \leq 4$$

and hence we are done, or $e_0 \left(1-\rho\right)^2 > 4$. In the latter case, we consider

$$v(x,t) = u\left(x_0 + e_0^{-\frac{1}{2}} x, t_0 + e_0^{-1} t\right),\ (x,t) \in P_1.$$

Then we have $e(v)(0) = 1$ and

$$\sup_{P_1} e(v) \leq e_0^{-1} \sup_{P_{1+\frac{\rho}{2}}} e(u) \leq e_0^{-1} \frac{e_0(1-\rho)^2}{(1-\frac{\rho}{2})^2} = 4.$$

Then by (5.3.3), v satisfies

$$|v_t - \Delta v| \le C\,|\nabla v| \quad \text{in } P_1.$$

Hence Lemma 5.3.4 implies that

$$1 = e(v)(0) \le C \int_{P_1} e(v) \le C \sup_{0\le t\le 1} \int_{B_1} |\nabla u|^2 \le C\epsilon_0,$$

which is impossible if we choose ϵ_0 to be sufficiently small. □

By Proposition 6.2.6, we conclude that $u \in C^\infty(M \setminus \mathcal{S}(u, T_0), N)$.

Step 6. Blow-up analysis near singular points. For simplicity, assume $(0,0)$ is a singular point of $u \in C^\infty(P_1(0,0) \setminus \{(0,0)\}, N)$ solving (5.3). Then there exist $r_k \downarrow 0$, $z_k = (x_k, t_k)$ with $x_k \to 0$, $t_k \to 0$ such that

$$E\left(u(t_k), B_{r_k}(x_k)\right) = \sup_{z=(x,t)\in P_1, -1\le t\le t_k} E\left(u(t), B_{r_k}(x)\right) = \frac{\epsilon_1}{C} \tag{6.35}$$

where $C > 0$ is a large number to be chosen. Assume $t_k - 4r_k^2 \ge -1$, define

$$v_k(x,t) = u\left(x_k + r_k x, t_k + r_k^2 t\right), \ (x,t) \in P_k$$

where $P_k = P_{r_k^{-1}}$ converges to $\mathbb{R}^2 \times \mathbb{R}_-$. Note that

$$\int_{P_k} \left|\frac{\partial v_k}{\partial t}\right|^2 \le \int_{t_k - r_k^2}^{t_k} \int_M |\partial_t u|^2 \to 0$$

$$E\left(v_k(t)\right) \le E(u_0), \ -r_k^{-2} \le t \le 0.$$

Moreover, we have

$$\sup_{(x,t)\in P_k} E\left(v_k(t), B_2(x)\right) \le C \sup_{(x,t)\in P_1} E\left(u(t), B_{r_k}(x)\right) \le \epsilon_1.$$

Therefore by proposition 6.2.6, we have that $v_k \in C^3_{\text{loc}}(\mathbb{R}^2 \times \mathbb{R}_-, N)$ is uniformly bounded for any $k \ge 1$. Hence we may assume $v_k \to \omega$ in $C^l_{\text{loc}}(\mathbb{R}^2 \times \mathbb{R}_-, N)$. Hence $\omega \in C^\infty(\mathbb{R}^2 \times (-\infty, 0), N)$ solves (5.3). Since $\omega_t \equiv 0$, $\omega \in C^\infty(\mathbb{R}^2, N)$ is a harmonic map satisfying

$$\epsilon_1 \le \int_{\mathbb{R}^2} e(\omega) < +\infty.$$

Therefore ω can be lifted to be a nontrivial harmonic map from S^2. We have now completed the proof of Theorem 6.2.1. □

We conclude this section with a few remarks.

Remark 6.2.7 One can apply Theorem 6.2.1 to give alternative proofs of Theorems 6.1.9 and 6.1.10 in the previous section. We refer interested readers to the articles [194, 196].

Remark 6.2.8 Theorem 6.2.1 has been extended by Chang [36] to the heat flow of harmonic maps from any Riemannian surface with ∂M under the Dirichlet boundary condition. Note that the theorem by Lemaire [114] on nonexistence of harmonic maps with finite energy from $\mathbb{R}^2_+$, with constant value on $\partial\mathbb{R}^2_+$, can be used to rule out possible bubbles near the boundary. Hence the solution obtained in [36] is smooth near the boundary. Ma [141] has extended Theorem 6.2.1 to the free boundary problem of the heat flow of harmonic maps from Riemannian surfaces with boundaries.

Remark 6.2.9 Freire [59, 60] has proved the uniqueness of weak solutions to the heat flow of harmonic maps in dimensions two in the class that the energy $E(u(t))$ is nonincreasing with respect to t. On the other hand, Topping [201] and Bertsch-Dal Passo-Van der Hout [13] have independently constructed weak solutions to the heat flow of harmonic maps in dimensions two that are different from Struwe's solution given by Theorem 6.2.1 by attaching reserve bubbles so that the energy $E(u(t))$ increase by a jump of 4π each time a bubble is attached.

6.3 Finite time singularity in dimension two

Without the curvature assumption on N, the short-time smooth solution may develop singularity in finite time even in dimensions two. Here we present the well-known example by Chang-Ding-Ye [23] on such a finite time singularity. This example also suggests that the global, weak solution to (5.3) and (5.4) by Struwe [193] is optimal in some sense.

Let $B_1(0) \subset \mathbb{R}^2$ be the unit ball with center at 0, consider equivariant maps $u_0 : B_1(0) \to S^2$:

$$u_0(r,\theta) = \left(e^{i\theta}\sin h_0(r), \cos h_0(r)\right) \quad (r,\theta) \in [0,1]\times[0,2\pi]$$

where $h : [0,1] \to \mathbb{R}$ is such that $h(0) = 0$ and $u_0(0,\cdot) = (0,0,1) \in S^2$. By direct calculation, the Dirichlet energy of u_0 is given by

$$\int_{B_1(0)} |\nabla u|^2 = 2\pi \int_0^1 \left(h_r^2 + \frac{\sin^2 h}{r^2}\right) r\,dr. \tag{6.36}$$

Let $u : B_1(0)\times[0,T) \to S^2$ be the corresponding smooth solution of (5.3)-(5.4), defined on a maximal time interval $[0,T)$. By uniqueness, u is also equivariant and can be written as

$$u(r,\theta,t) = \left(e^{i\theta}\sin h(r,t), \cos h(r,t)\right) \quad (r,\theta,t) \in [0,1]\times[0,2\pi]\times[0,T).$$

It is easy to verify that (5.3) and (5.4) can be written as

$$h_t - h_{rr} - \frac{1}{r}h_r - \frac{\sin 2h}{2r^2} = 0,\ 0 \le r \le 1, t > 0 \tag{6.37}$$
$$h(r,0) = h_0(r),\ 0 \le r \le 1 \tag{6.38}$$
$$h(0,t) = h_0(0)(=0),\ t \ge 0 \tag{6.39}$$
$$h(1,t) = h_0(1) := b,\ t \ge 0. \tag{6.40}$$

Note that the stereographic projection from the south pole maps $B_1(0)$ to the upper half sphere S^2_+ is given by

$$\frac{\sin h}{1+\cos h} = r,$$

that is

$$h(r) = \arccos\left(1 - \frac{r^2}{1+r^2}\right).$$

Composing with a dilation $r \to \frac{r}{\lambda}$, we get a family

$$\phi_\lambda(r) = \arccos\left(\frac{\lambda^2 - r^2}{\lambda^2 + r^2}\right), \lambda > 0,$$

which solves

$$(\phi_\lambda)_{rr} + \frac{1}{r}(\phi_\lambda)_r - \frac{\sin(2\phi_\lambda)}{2r^2} = 0, \ 0 \le r \le 1. \tag{6.41}$$

Chang-Ding-Ye [23] have proved the following theorem.

Theorem 6.3.1 *For $|b| > \pi$, the solution h to (6.37)-(6.40) blows up in finite time.*

Proof. First note that (6.36) implies

$$\int_0^1 |h_r|^2 (r,t) r\, dr \le E(u_0). \tag{6.42}$$

Hence by Sobolev embedding, $h(\cdot, t)$ is locally Hölder continuous on $(0,1]$ uniformly in t, and a singularity can only develop at the origin.
Assume $b > \pi$, the key ingredient is to construct a subsolution of (6.37)-(6.40) with $f(0,t) = 0 \le f \le f(1,t) = b$ such that $f_r(0,t) \to \infty$ as $t \to T$ for some $T < +\infty$. In fact, let $h_0(r) = f(r,0)$ and h be the corresponding solution of (4.2)-(4.5). Then the maximum principle implies $h \ge f$ on $[0,1] \times [0,T)$. Hence h must blow up no later than T.
Let

$$f(r,t) = \phi_{\lambda(t)}(r) + \phi_\mu(r^{1+\epsilon}),$$

where $\epsilon > 0$, $\mu > 0$ and $\lambda(t)$ will be chosen suitably. Since

$$\phi_\mu\left(r^{1+\epsilon}\right) = \arccos\left(\frac{\mu^2 - r^{2+2\epsilon}}{\mu^2 + r^{2+2\epsilon}}\right) \to 0, \text{ as } \mu \to \infty$$

uniformly in $r \in [0,1]$. Hence for any $\epsilon > 0$, there is $\mu = \mu(\epsilon) > 0$ such that

$$\cos\left(\phi_\mu(r^{1+\epsilon})\right) \ge \frac{1}{1+\epsilon}, \ r \in [0,1]. \tag{6.43}$$

Set $\theta(r) = \phi_\mu(r^{1+\epsilon})$. Then one can check

$$\theta_{rr} + \frac{1}{r}\theta_r - \frac{(1+\epsilon)^2 \sin 2\theta}{2r^2} = 0, \ 0 \le r \le 1. \tag{6.44}$$

Now we compute the tension field of f:

$$\begin{aligned}\tau(f) &:= f_{rr} + \frac{1}{r} f_r - \frac{\sin 2f}{2r^2} \qquad (6.45)\\ &= \frac{\sin 2\phi_\lambda - \sin 2(\phi_\lambda + \theta) + (1+\epsilon)^2 \sin 2\theta}{2r^2}\\ &= \frac{(1+\epsilon)^2 \cos\theta \sin\theta - \cos(2\phi_\lambda + \theta)\sin\theta}{r^2}\\ &\geq \frac{(1+\epsilon) - \cos(2\phi_\lambda + \theta)}{r^2} \sin\theta\\ &\geq \frac{\epsilon}{r^2}\sin\theta\\ &= \frac{\epsilon}{r^2}\frac{2\mu r^{1+\epsilon}}{\mu^2 + r^{2+2\epsilon}}\\ &\geq \epsilon_1 r^{\epsilon-1}, \qquad (6.46)\end{aligned}$$

where $\epsilon_1 = \frac{2\mu\epsilon}{\mu^1+1} > 0$. On the other hand, we have

$$f_t = -\frac{2r}{\lambda^2 + r^2}\lambda'(t).$$

Let $\lambda'(t) = -\delta\lambda^\epsilon$, $\delta > 0$ to be chosen. Solving this ODE gives

$$\lambda(t) = \left(\lambda_0^{1-\epsilon} - (1-\epsilon)\,\delta t\right)^{\frac{1}{1-\epsilon}}. \qquad (6.47)$$

By direct calculation, we have

$$f_r(0,t) = -\frac{2}{\lambda(t)} \to \infty$$

when $t \uparrow T = \frac{\lambda_0^{1-\epsilon}}{(1-\epsilon)\delta}$, as $\lambda(T) = 0$.
Finally, we have

$$f_t - \tau(f) \leq \frac{2\delta\lambda^\epsilon r}{\lambda^2 + r^2} - \epsilon_1 r^{\epsilon-1} = \left(\frac{2\delta\lambda^\epsilon r^{2-\epsilon}}{\lambda^2 + r^2} - \epsilon_1\right) r^{\epsilon-1} \leq 0$$

provided that we choose sufficiently small $\delta = \delta(\epsilon) > 0$. Here we have used Young's inequality:

$$\lambda^\epsilon r^{2-\epsilon} \leq C(\epsilon)(\lambda^2 + r^2).$$

Observe that if we choose λ_0 sufficiently small and μ sufficiently large, then $|f(r,0) - \pi|$ and $|f(1,t) - \pi|$ can be made arbitrarily small. In particular, $f(r,t)$ is a subsolution of (4.2)-(4.5). Hence $h(r,t) \geq f(r,t)$ and $\lim_{t\uparrow T} h_r(0,t) = +\infty$ so that (6.37)-(6.40) blows up at finite time. □

Remark 6.3.2 For $|b| \leq \pi$, Grayson-Hamilton [62] and Chang-Ding [22] have independently proved that (6.37)-(6.40) has a global, smooth solution. Hence in this sense Theorem 6.3.1 is optimal.

6.4 Bubbling phenomena for heat flows in dimension two

From the discussion in the previous two sections, we know that in dimension two the energy concentration leads to both the failure of strong convergence and the formation of singularity for both harmonic maps and their heat flows. The central question here is (i) the energy identity, which asks whether the loss of energy during the convergence process can be recovered by a finite number of harmonic S^2's, and (ii) the bubble tree convergence, which asks whether the sequence converges continuously to the limiting map which is formed by gluing finitely many harmonic S^2's to a weak limiting map. In the last several years, there have been many works done in this direction. We refer the reader to the important works by Jost [103], Parker [152] on harmonic maps from surfaces, by Parker-Wolfson [153] on pseudo holomorphic curves, and Qing [157], Ding-Tian [43], Wang [206] on the energy identity, and Qing-Tian [158], and Lin-Wang [133] on bubble tree convergence for approximate harmonic maps with tension fields bounded in L^2 in dimensions two.

In this section, we will analyze the behavior near a singularity of an almost smooth heat flow of harmonic maps from a Riemannian surface obtained in Theorem 6.2.1. We will establish both bubble tree convergence at $t = +\infty$ and the energy identity at a finite time singular point for the class of heat flows obtained in Theorem 6.2.1. Our presentation here follows [133] very closely.

We start by stating two main theorems of this section. Throughout this section, assume that $u : M \times \mathbb{R}_+ \to N$ is the solution obtained by Theorem 6.2.1. Let $t_n \to \infty$ be such that

$$\lim_{n\to\infty} \|\partial_t u(t_n)\|_{L^2(M)} = 0 \text{ and } \lim_{n\to\infty} \int_{M\times[t_n-1,t_n]} |\partial_t u|^2 = 0. \tag{6.48}$$

Theorem 6.4.1 *There exist a harmonic map $u_\infty \in C^\infty(M,N)$ and a finite number of bubbles $\{\omega_i\}_{i=1}^m$, $\{a_n^i\}_{i=1}^m \subset M$, and $\{\lambda_n^i\}_{i=1}^m \subset \mathbb{R}_+$ such that*

$$\left\| u(t_n) - u_\infty - \sum_{i=1}^m \omega_n^i \right\|_{L^\infty(M)} \to 0, \tag{6.49}$$

where

$$\omega_n^i(\cdot) = \omega_i\left(\frac{\cdot - a_n^i}{\lambda_n^i}\right) - \omega_i(\infty).$$

Theorem 6.4.2 *For $0 < T_0 < +\infty$, let $u \in C^\infty(M \times (0,T_0), N)$ solve (5.3) with T_0 as its singular time. Then there exist finitely many bubbles $\{\omega_i\}_{i=1}^l$ such that*

$$\lim_{t\uparrow T_0} E(u(t)) = E(u(T_0)) + \sum_{i=1}^l E\left(\omega_i, S^2\right). \tag{6.50}$$

There are two main ingredients in the proof of Theorem 6.4.1: (i) an almost convexity estimate of the angular energy, and (ii) the Pohozaev type inequality that controls the radial energy by the angular energy.

Lemma 6.4.3 *There exists $\epsilon_0 > 0$ such that if $u \in C^\infty([T_1, T_2] \times S^1, N)$ satisfies*

$$u_{tt} + u_{\theta\theta} = A(u)\,(\nabla u, \nabla u) + F, \tag{6.51}$$

and

$$\sup_{[T_1,T_2]\times S^1} |\nabla u| \le \epsilon_0.$$

Then there is $C > 0$ such that for $t \in [T_1, T_2]$,

$$\frac{d^2}{dt^2}\int_{S^1} |u_\theta|^2 \ge \int_{S^1} |u_\theta|^2 - C\int_{S^1} |F|^2. \tag{6.52}$$

Proof. Direct computation, integration by parts, and using (6.51) gives

$$\begin{aligned}
\frac{d^2}{dt^2}\int_{S^1} |u_\theta|^2 &= 2\int_{S^1} |u_{\theta t}|^2 + 2\int_{S^1} \langle u_\theta, u_{\theta tt}\rangle \\
&= 2\int_{S^1} |u_{\theta t}|^2 - 2\int_{S^1} \langle u_{\theta\theta}, u_{tt}\rangle \\
&= 2\int_{S^1} |u_{\theta t}|^2 + 2\int_{S^1} |u_{\theta\theta}|^2 \\
&\quad -2\int_{S^1} \langle u_{\theta\theta}, A(u)\,(\nabla u, \nabla u) + F\rangle \\
&= I + II + III.
\end{aligned}$$

We estimate III as follows.

$$\begin{aligned}
III &= 2\int_{S^1} \langle u_\theta, (A(u)\,(\nabla u, \nabla u))_\theta\rangle - 2\int_{S^1} \langle u_{\theta\theta}, F\rangle \\
&= 2\int_{S^1} \langle u_\theta, (\nabla A(u)\,(\nabla u, \nabla u)\, u_\theta + 2A(u)\,(u_{\theta\theta}, u_\theta) \\
&+ 2A(u)(u_{\theta t}, u_t))\rangle - 2\int_{S^1} \langle u_{\theta\theta}, F\rangle .
\end{aligned}$$

Hence by the Cauchy-Schwarz inequality

$$\begin{aligned}
|III| &\le 2\,\|\nabla A\|_{L^\infty(N)} \sup_{[T_1,T_2]\times S^1} |\nabla u|^2 \int_{S^1} |u_\theta|^2 \\
&+ 4\,\|A\|_{L^\infty(N)} \int_{S^1} |u_{\theta\theta}|\,|u_\theta|^2 \\
&+ 4\,\|A\|_{L^\infty(N)} \int_{S^1} |u_{\theta t}|\,|u_\theta|\,|u_t| + 2\int_{S^1} |u_{\theta\theta}|\,|F| \\
&\le \left(\frac{1}{2} + C\epsilon_0^2\right)\int_{S^1} |u_{\theta\theta}|^2 + C\epsilon_0^2\int_{S^1} |u_\theta|^2 + C\epsilon_0^2\int_{S^1} |u_{\theta t}|^2 + C\int_{S^1} |F|^2 .
\end{aligned}$$

Therefore if we choose ϵ_0 sufficiently small, then

$$\frac{d^2}{dt^2}\int_{S^1} |u_\theta|^2 \ge \frac{17}{16}\int_{S^1} |u_{\theta\theta}|^2 - \frac{1}{16}\int_{S^1} |u_\theta|^2 - C\int_{S^1} |F|^2 .$$

On the other hand, the Poincaré inequality on S^1 gives

$$\int_{S^1} |u_\theta|^2 \leq \int_{S^1} |u_{\theta\theta}|^2 .$$

Therefore

$$\frac{d^2}{dt^2}\int_{S^1} |u_\theta|^2 \geq \int_{S^1} |u_\theta|^2 - C\int_{S^1} |F|^2 .$$

This gives (6.52). □

Now we analyze the solutions to the following 2nd ODE.

$$\begin{aligned} P_1'' - P_1 &= -G(t), T_1 \leq t \leq T_2, && (6.53)\\ P_1(T_1) &= \epsilon_1, && (6.54)\\ P_1(T_2) &= \epsilon_2, && (6.55) \end{aligned}$$

where $G(\geq 0) \in L^1([T_1, T_2])$ is given, $\epsilon_1 = \int_{S^1\times\{T_1\}} |u_\theta|^2$, and $\epsilon_2 = \int_{S^1\times\{T_2\}} |u_\theta|^2$. In fact, we can solve (6.53)-(6.55) explictly and get

Lemma 6.4.4 *Let $P_1 : [T_1, T_2] \to \mathbb{R}$ be a solution to (6.53)-(6.55). Then*

$$P_1(t) = Ae^t + Be^{-t} - \frac{1}{2}\int_t^{T_2} G(s)\left(e^{s-t} - e^{t-s}\right)\, ds, \tag{6.56}$$

where

$$A = \frac{e^{T_2}\epsilon_2 - e^{T_1}\epsilon_1 + \frac{1}{2}\int_{T_1}^{T_2} G(s)\left(e^s - e^{2T_1 - s}\right)\, ds}{e^{2T_2} - e^{2T_1}},$$

$$B = \frac{e^{T_1+2T_2}\epsilon_1 - e^{2T_1+T_2}\epsilon_2}{e^{2T_2} - e^{2T_1}} - \frac{1}{2}e^{2T_2}\frac{\int_{T_1}^{T_2} G(s)\left(e^s - e^{2T_1-s}\right)\, ds}{e^{2T_2} - e^{2T_1}}.$$

For $T_1 \leq t \leq T_2$, denote $P(t) = \int_{S^1\times\{t\}} |u_\theta|^2$. Then the maximum principle implies

$$P(t) \leq P_1(t), \ \forall t \in [T_1, T_2]. \tag{6.57}$$

Hence we obtain

Lemma 6.4.5 *Assume that $G(t) = e^{-2t}H(t)$ with $H \in L^1([T_1, T_2])$ and $0 < T_1 << T_2 < \infty$. Then*

$$\begin{aligned} \int_{T_1}^{T_2} |P(t)|^{\frac{1}{2}}\, dt &\leq |A|^{\frac{1}{2}}\left(e^{\frac{T_2}{2}} - e^{\frac{T_1}{2}}\right) + |B|^{\frac{1}{2}}\left(e^{-\frac{T_1}{2}} - e^{-\frac{T_2}{2}}\right)\\ &+ \left(e^{-\frac{T_1}{2}} - e^{-\frac{T_2}{2}}\right)\left(\int_{T_1}^{T_2} |H(t)|\, dt\right)^{\frac{1}{2}}\\ &\leq C\left(\sqrt{\epsilon_1} + \sqrt{\epsilon_2}\right) + C\left(\int_{T_1}^{T_2} |H(t)|\, dt\right)^{\frac{1}{2}}. \end{aligned} \tag{6.58}$$

Now we derive the Pohozaev inequality for two dimensional approximate harmonic maps in dimensions two.

Lemma 6.4.6 *Let* $u \in W^{2,2}(B_1^2, N)$ *solve*

$$\Delta u + A(u)(\nabla u, \nabla u) = h, \quad \text{with } h \in L^2(B_1^2).$$

Then

$$\int_{\partial B_R} \left|\frac{\partial u}{\partial r}\right|^2 \leq R^{-2} \int_{\partial B_R} |u_\theta|^2 + 2\int_{B_R} |h|\,|\nabla u|\,, \tag{6.59}$$

for any $0 < R < 1$.

Proof. Multiplying both sides of the equation of u by $x \cdot \nabla u$ and integrating over B_R, we get

$$\int_{B_R} |\nabla u|^2 - R\int_{\partial B_R} |u_r|^2 + \frac{1}{2}\int_{B_R} x \cdot \nabla\left(|\nabla u|^2\right) = -\int_{B_R} \langle h, x \cdot \nabla u\rangle\,.$$

Note that

$$\frac{1}{2}\int_{B_R} x \cdot \nabla\left(|\nabla u|^2\right) = -\int_{B_R} |\nabla u|^2 + \frac{1}{2}R\int_{\partial B_R} |\nabla u|^2\,.$$

Hence

$$\frac{1}{2}\int_{\partial B_R} |\nabla u|^2 - \int_{\partial B_R} |u_r|^2 = -R^{-1}\int_{B_R} \langle h, x \cdot \nabla u\rangle\,,$$

which implies (6.59), since $|\nabla u|^2 = |u_r|^2 + \frac{1}{r^2}|u_\theta|^2$. □

Lemma 6.4.7 *Let* $u \in C^\infty(B_1^2 \times [0, t_0), N)$ *solve (5.3). Then for* $0 < t \leq s < t_0$ *and* $0 < R \leq \frac{1}{2}$,

$$\int_{B_R} |\nabla u|^2 (x, s)\, dx \leq \int_{B_{2R}} |\nabla u|^2 (x, t)\, dx + C\,(s - t)\,\frac{E_0}{R^2}, \tag{6.60}$$

and

$$\int_{B_R} |\nabla u|^2 (x, t)\, dx \leq \int_{B_{2R}} |\nabla u|^2 (x, s)\, dx + C\int_t^s \int_{B_1} |\partial_t u|^2 + C\,(s - t)\,\frac{E_0}{R^2}, \tag{6.61}$$

where $E_0 = E(u(\cdot, 0))$.

Proof. Let $\phi \in C_0^\infty(B_1^2)$ be such that $0 \leq \phi \leq 1$, $\phi = 1$ on B_R, and $\phi = 0$ outside B_{2R}. Multiplying (5.3) by $\phi^2 \partial_t u$ gives

$$\begin{aligned}
&-\left(2\int_{B_1^2} |\nabla u|^2 |\nabla \phi|^2 + \frac{1}{2}\int_{B_1^2} |\partial_t u|^2 \phi^2\right) \\
&\leq \int_{B_1^2} |\partial_t u|^2 \phi^2 + \frac{d}{dt}\left(\frac{1}{2}\int_{B_1^2} |\nabla u|^2 \phi^2\right) \\
&\leq\ 2\int_{B_1^2} |\nabla u|^2 |\nabla \phi|^2 + \frac{1}{2}\int_{B_1^2} |\partial_t u|^2 \phi^2.
\end{aligned}$$

Integrating these two inequalities from t to s yields (6.60) and (6.61). □

Now we need a refined version of the small energy regularity estimate, which is a consequence of Proposition 6.2.6 and Lemma 6.4.7 (see also §7 below).

Lemma 6.4.8 *There exist $\epsilon_0 > 0$ and $C > 0$ such that if $u \in C^\infty(\mathbb{R}^2 \times \mathbb{R}_-, N)$ is a solution to (5.3) satisfying*

$$R^{-2} \int_{P_R(z)} |\nabla u|^2 \leq \epsilon_0^2,$$

for some $z \in \mathbb{R}^2 \times \mathbb{R}_-$. Then

$$R^2 \sup_{P_{\frac{R}{4}}(z)} |\nabla u|^2 \leq CR^{-2} \int_{P_R(z)} |\nabla u|^2, \tag{6.62}$$

and

$$R^4 \sup_{P_{\frac{R}{2}}(z)} |\partial_t u|^2 \leq C(\epsilon_0). \tag{6.63}$$

Proof. From the proof of Lemma 6.4.7, it is easy to see that

$$\int_{B_{\frac{R}{2}}(x)} |\nabla u(s)|^2 \leq R^{-2} \int_{P_R(z)} |\nabla u|^2 \text{ for all } s \in [t - \frac{R^2}{4}, t).$$

Hence the condition of Proposition 6.2.6 is satisfied, and the conclusions follow. □

Proof of Theorem 6.4.1:
For simplicity, we may assume $M = B_1^2$. Let $t_n \uparrow \infty$ be such that (6.48) holds. Denote $u_n = u(t_n)$. From the standard reduction on the number of bubbles (cf. [157, 206]), Theorem 6.4.1 follows by dealing with a single bubble.
Assume that for $\delta > 0$ small, $u_n \to u_\infty$ in $H^1(B_\delta \setminus \{0\}, N)$ locally but not in $H^1(B_\delta, N)$. Also assume that there exists only one bubble ω_1 such that for some $\lambda_n \downarrow 0$ and $x_n \to 0$,

$$\tilde{u}_n(x) = u_n(x_n + \lambda_n x) \to \omega_1 \text{ in } H^1_{\text{loc}} \cap C^1_{\text{loc}}(\mathbb{R}^2, N).$$

For large $R > 0$, denote

$$A_n(\delta, R) = \left\{ x \in \mathbb{R}^2 \mid R\lambda_n \leq |x - x_n| \leq \delta \right\},$$

and

$$\Sigma_n(\delta, R) = [|\log \delta|, |\log R\lambda_n|] \times S^1.$$

Note that $f(r, \theta) = (e^{-r}, \theta) : \Sigma_n(\delta, R) \to A_n(\Sigma, R)$ is conformal provided that $\Sigma_n(\delta, R)$ is equipped with the flat metric. Let $v_n : \Sigma_n(\delta, R) \to N$ be defined by $v_n(r, \theta) = u_n(e^{-r}, \theta)$. Then

$$\Delta v_n + A(v_n)(\nabla v_n, \nabla v_n) = \bar{h}_n \text{ in } \Sigma_n(\delta, R), \tag{6.64}$$

where $\bar{h}_n(r,\theta) = e^{-2r}\partial_t u\,(e^{-r},\theta,t_n)$ satisfies

$$\left\|\bar{h}_n\right\|_{L^2([r,\infty)\times S^1)} \le e^{-r}\left\|\partial_t u(\cdot,t_n)\right\|_{L^2(B_{e^{-r}})}. \tag{6.65}$$

The conformal invariance of E implies,

$$\int_{\Sigma_n(\delta,R)} |\nabla v_n|^2 = \int_{A_n(\delta,R)} |\nabla u_n|^2. \tag{6.66}$$

From the assumption that there exists only one bubble ω_1, we also have (cf. [43, 157, 206])

$$\int_{B_{e^{-(r-2)}}\setminus B_{e^{-(r+2)}}} |\nabla u_n|^2 = \int_{[r-2,r+2]\times S^1} |\nabla v_n|^2 \le \frac{1}{4}\epsilon_0^2, \tag{6.67}$$

for all $r \in [|\log\delta|, |\log R\lambda_n|]$.

Now we have the following lemma.

Lemma 6.4.9 *Assume u_n and v_n are as above. Then*

$$\lim_{\delta\downarrow 0}\lim_{R\uparrow\infty}\lim_{n\to\infty}\int_{A_n(\delta,R)} |\nabla u_n|^2 = 0, \tag{6.68}$$

and

$$\lim_{\delta\downarrow 0}\lim_{R\uparrow\infty}\lim_{n\to\infty} osc_{A_n(\delta,R)} u_n = 0. \tag{6.69}$$

Proof. From (6.67) and (6.48), one can apply Lemma 6.4.8 to get

$$\begin{aligned} |\nabla v_n|(r,\theta) &= e^{-r}|\nabla u_n|(e^{-r},\theta) \le C(\epsilon_0), \\ \bar{h}_n(r,\theta) &= e^{-2r}|\partial_t u|(e^{-r},\theta,t_n) \le C(\epsilon_0) \end{aligned} \tag{6.70}$$

for all $r \in [|\log\delta|, |\log R\lambda_n|]$. Let $G_n(r) = \int_{S^1\times\{r\}}\left|\bar{h}_n(r,\theta)\right|^2$. Then we have

$$\int_{|\log\delta|}^{|\log R\lambda_n|} e^{2r} G_n(r)\,dr = \int_{A_n(\delta,R)} |\partial_t u(t_n)|^2 \to 0.$$

Using the $W^{2,4}$-estimate, we get

$$\begin{aligned} \left\|\nabla^2 v_n\right\|_{L^4([r-1,r+1]\times S^1)} &\le C(\|\nabla v_n\|_{L^4([r-2,r+2]\times S^1)} \\ &+ \left\||\nabla v_n|^2\right\|_{L^4([r-2,r+2]\times S^1)} + \left\|\bar{h}_n\right\|_{L^4([r-2,r+2]\times S^1)}) \\ &\le C(\|\nabla v_n\|^{\frac{1}{2}}_{L^\infty(\Sigma_n(\delta,R))}\|\nabla v_n\|^{\frac{1}{2}}_{L^2([r-2,r+2]\times S^1)} \\ &+ \left\|\bar{h}_n\right\|^{\frac{1}{2}}_{L^\infty(\Sigma_n(\delta,R))}\left\|\bar{h}_n\right\|^{\frac{1}{2}}_{L^2([r-2,r+2]\times S^1)}) \le C\epsilon_0. \end{aligned}$$

for all $r \in [|\log\delta|, |\log R\lambda_n|]$.
Therefore by the Sobolev embedding theorem, we have

$$\|\nabla v_n\|_{L^\infty(\Sigma_n(\delta,R))} \le C\epsilon_0.$$

Hence we can apply Lemma 6.4.5 with u, F, G, T_1, T_2, replaced by v_n, $\bar{h}_n$, G_n, $|\log\delta|$, $|\log R\lambda_n|$ respectively, to conclude

$$\begin{aligned}\int_{|\log\delta|}^{|\log R\lambda_n|}\left(\int_{S^1}|(v_n)_\theta|^2\right)^{\frac{1}{2}} &\le \left(\int_{S^1\times\{|\log R\lambda_n|\}}|(v_n)_\theta|^2\right)^{\frac{1}{2}}\\ &+ \left(\int_{S^1\times\{|\log\delta|\}}|(v_n)_\theta|^2\right)^{\frac{1}{2}}\\ &+ \sqrt{\delta}\left(\int_{B_\delta}|\partial_t u(t_n)|^2\right)^{\frac{1}{2}}\to 0. \qquad (6.71)\end{aligned}$$

Here we have used the fact that

$$\max\left\{\int_{S^1\times\{|\log R\lambda_n|\}}|(v_n)_\theta|^2,\quad \int_{S^1\times\{|\log\delta|\}}|(v_n)_\theta|^2\right\}\to 0,\ \text{as } n\to\infty.$$

Applying Lemma 6.4.6, we get

$$\int_{S^1\times\{r\}}|(v_n)_r|^2\le\int_{S^1\times\{r\}}|(v_n)_\theta|^2+2e^{-r}\int_{B_{e^{-r}}}|\partial_t u(t_n)|\,|\nabla u_n|\,,$$

for any $r\in[|\log\delta|,|\log R\lambda_n|]$. In particular,

$$\begin{aligned}&\int_{|\log\delta|}^{|\log R\lambda_n|}\left(\int_{S^1}|(v_n)_r|^2\right)^{\frac{1}{2}}\\ \le\ &\int_{|\log\delta|}^{|\log R\lambda_n|}\left(\int_{S^1}|(v_n)_\theta|^2\right)^{\frac{1}{2}}\\ +\ &2\int_{|\log\delta|}^{|\log R\lambda_n|}e^{-\frac{r}{2}}\left(\int_{B_{e^{-r}}}|\partial_t u(t_n)|\,|\nabla u_n|\right)^{\frac{1}{2}}\\ \le\ &o(1)+2\left(\int_{|\log\delta|}^{|\log R\lambda_n|}e^{-\frac{r}{2}}\right)\cdot\left(\int_{B_\delta}|\partial_t u(t_n)|^2\right)^{\frac{1}{4}}\\ \cdot\ &\left(\int_{B_\delta}|\nabla u_n|^2\right)^{\frac{1}{4}}\\ \le\ &o(1)+2\sqrt{\delta}\left(\int_{B_\delta}|\partial_t u(t_n)|^2\right)^{\frac{1}{4}}\left(\int_{B_\delta}|\nabla u_n|^2\right)^{\frac{1}{4}}\to 0.\end{aligned}$$

Therefore

$$\int_{\Sigma_n(\delta,R)}|\nabla v_n|\le(2\pi)^{\frac{1}{2}}\int_{|\log\delta|}^{|\log R\lambda_n|}\left((\int_{S^1}|(v_n)_r|^2)^{\frac{1}{2}}+(\int_{S^1}|(v_n)_\theta|^2)^{\frac{1}{2}}\right)dr\to 0.$$

This clearly implies (6.69). It is easy to see that (6.68) follows from (6.69). □

In order to prove Theorem 6.4.2, we need the following lemma.

Lemma 6.4.10 *Let $u \in C^\infty(B_1^2 \times (0,t_0), N)$ solve (5.3) with $(0,t_0)$ being its only singular point. Then there exists a constant $m > 0$ such that*

$$|\nabla u|^2(x,t)\,dx \to m\delta_0 + |\nabla u|^2(x,t_0)\,dx, \tag{6.72}$$

for $t \uparrow t_0$, as convergence of Radon measures. Here δ_0 denotes the Dirac mass at 0.

Proof. For any two sequences $s_i \uparrow t_0$, $t_i \uparrow t_0$, according to Lemma 6.4.7 there exist two constants $m > 0$ and $m' > 0$ such that, up to subsequences,

$$\begin{aligned} |\nabla u|^2(x,s_i)\,dx &\to m\delta_0 + |\nabla u|^2(x,t_0)\,dx \\ |\nabla u|^2(x,t_i)\,dx &\to m'\delta_0 + |\nabla u|^2(x,t_0)\,dx, \end{aligned}$$

as convergence of Radon measures in B_1^2.

For any $\epsilon > 0$, there exists $\eta > 0$ such that $\int_{B_{2\eta}^2} |\nabla u|^2(x,t_0) \le \epsilon$. Therefore, we have

$$\begin{aligned} m &\ge \int_{B_{2\eta}^2} |\nabla u|^2(x,s_i) - \epsilon \\ &\ge \int_{B_\eta} |\nabla u|^2(x,t_i) - C\delta^{-2}|s_i - t_i|E_0 - \int_{s_i}^{t_i}\int_{B_1^2} |\partial_t u|^2 - \epsilon \\ &\ge \int_{B_\eta} |\nabla u|^2(x,t_i) - 2\epsilon \ge m' - 2\epsilon. \end{aligned}$$

Hence $m \ge m'$. Similarly $m \le m'$. □

Proof of Theorem 6.4.2:
Assume $T_0 = 0$, $M = B_1^2$, and $(0,0)$ is the singular point of u. By (6.72), there exist $t_n \uparrow 0$ and $\lambda_n \downarrow 0$ such that

$$\lim_{n\to\infty} \int_{B_{\lambda_n}} |\nabla u|^2(x,t_n)\,dx = m.$$

Let $u_n(x,t) = u(\lambda_n x, t_n + \lambda_n^2 t)$. Then u_n satisfies (5.3) on $B^2_{\lambda_n^{-1}} \times [-2,0)$, and

$$\int_{-2}^{2}\int_{B^2_{\lambda_n^{-1}}} |\partial_t u_n|^2 = \int_{t_n-2\lambda_n^2}^{t_n+2\lambda_n^2}\int_{B_1^2} |\partial_t u|^2 \to 0, \quad \text{as } n\to\infty.$$

Therefore, by Fubin's theorem, there exists $\eta_n \in (-1, -\frac{1}{2})$ such that

$$\int_{B^2_{\lambda_n^{-1}}} |\partial_t u_n|^2(\cdot,\eta_n) \to 0, \quad \int_{B^2_{\lambda_n^{-1}}\times(-2,2)} |\partial_t u_n|^2 \to 0.$$

Note also that

$$\int_{B_R} |\nabla u_n|^2(\cdot,\eta_n) \ge \int_{B_1} |\nabla u_n|^2(\cdot,0) - CR^{-2}E_0 \ge m - CR^{-2}E_0.$$

In particular,

$$\lim_{R\to\infty}\int_{B_R}|\nabla u_n|^2(\cdot,\eta_n)\geq m.$$

In fact, by Lemma 6.72 we have

$$\lim_{R\to\infty}\int_{B_R}|\nabla u_n|^2(\cdot,\eta_n)=m.$$

Hence for each $R>0$ $u_n(\cdot,\eta_n)$ converges weakly to $v\in H^1(B_R,N)$ and v is a constant, since we can assume $|t_n|\leq 2\lambda_n^2$ and have

$$\int_{B_R}|u_n(\cdot,\eta_n)-u_n(\cdot,-t_n\lambda_n^{-2})|^2\leq 4\int_{-2}^{2}\int_{B_R}|\partial_t u_n|^2\to 0,$$

and

$$\int_{B_R}|\nabla u_n(\cdot,-t_n\lambda_n^{-2})|^2=\int_{B_{R\lambda_n}}|\nabla u|^2(\cdot,0)\to 0.$$

For each $R>0$, we now apply the proof of Theorem 6.4.2 to $u_n(\cdot,\eta_n)$ on B_R to conclude that there exist N_R bubbles $\{\omega_{i,R}\}_{i=1}^{N_R}$ such that

$$\lim_{n\to\infty}\int_{B_R}|\nabla u_n|^2(\cdot,\eta_n)=\sum_{i=1}^{N_R}E(\omega_{i,R},S^2).$$

Since there exists a universal $\epsilon_0>0$ such that any bubble $\omega:S^2\to N$ has $E(\omega,S^2)\geq\epsilon_0$, we have that $1\leq N_R\leq[\frac{m}{\epsilon_0}]$. Therefore, there are $d\in[1,[\frac{m}{\epsilon_0}]]$ and a subsequence $R\uparrow\infty$ such that $N_R=d$ and

$$m=\lim_{R\uparrow\infty}\lim_{n\to\infty}\int_{B_R}|\nabla u_n|^2(\cdot,\eta_n)=\lim_{R\uparrow\infty}\sum_{i=1}^{d}E(\omega_{i,R},S^2).$$

Note that for $i=1,\cdots,d$, $\{\omega_{i,R}\}$ are sequences of harmonic maps from S^2 to N whose energies are uniformly bounded. Hence we can apply the theorems on harmonic maps by [103] and [152] to conclude that for $i=1,\cdots,d$, there exist $N_i\in[1,[\frac{m}{\epsilon_0}]]$ and N_i bubbles $\{\omega_{i,j}\}_{j=1}^{N_i}$ such that

$$\lim_{R\uparrow\infty}E(\omega_{i,R},S^2)=\sum_{j=1}^{N_l}E(\omega_{i,j},S^2).$$

Therefore we have

$$m=\sum_{i=1}^{d}\sum_{j=1}^{N_i}E(\omega_{i,j},S^2),$$

where $\omega_{i,j}$ are bubbles for $1\leq i\leq d, 1\leq j\leq N_i$. The proof is complete. □

Remark 6.4.11 For the heat flow of harmonic maps from surfaces, it is an important question to ask whether the solution $u(t)$ converges at $t=+\infty$ and the uniqueness of the locations of the bubbles of the heat flow at $t = +\infty$. Topping [202, 203] has made some interesting progresses on this problem. For example, in [203], Topping has proved that for $M = N = S^2$, if for some $t_i \uparrow \infty$, a weak limit map u_∞ and all possible bubbles ω_j, $1 \le j \le K$, of the sequence $u(t_i) : S^2 \to S^2$ are either all holomorphic or all anti-holomorphic, then u_∞ is independent of $t_i \uparrow +\infty$. In this case, Topping has essentially proved the quantization estimate: $|E(u(t_i) - 4\pi k| \le C\|\partial_t u(t_i)\|^2_{L^2(S^2)}$ for i sufficiently large. In [203], he has obtained some further refine results along this line.

6.5 Approximate harmonic maps in dimension two

For both the energy identity and no neck formation for approximate harmonic maps from Riemannian surfaces, the assumption that the tension field is bounded in L^2 seems to be necessary for various methods developed by [157] [43] [206] [158] and [133] mentioned in the previous section. However, this condition is not conformally invariant. Parker [152] has constructed a sequence of approximate harmonic maps from a Riemannian surface with L^1 bounded tension fields, in which the energy identity fails. Hence a natural question to ask is whether the energy identity holds for approximate harmonic maps with tension fields bounded in L^p for $1 < p < 2$ in dimensions two. In this section, we present a result, due to Lin and Wang [135], which gives a confirmative answer to this question. In fact, the condition on the tension field in [135] (see Proposition 6.5.3 below) is essentially optimal.

Theorem 6.5.1 *Let M be a Riemannian surface without boundary. For any $p > 1$, assume that $\{u_i\} \subset H^1(M, S^{L-1})$ are such that the tension fields:*

$$\tau(u_i) \equiv \Delta u_i + |\nabla u_i|^2 u_i$$

are bounded in $L^p(M)$. If u_i converges to u weakly in $H^1(M, S^{L-1})$, then there exist finitely many harmonic S^2's, $\{\omega_j\}_{j=1}^l$, $\{a_i^j\}_{j=1}^l \subset M$, $\{\lambda_i^j\}_{j=1}^l \subset \mathbb{R}_+$ such that

$$\lim_{i\to\infty} \left\| u_i - u - \sum_{j=1}^l \omega_i^j \right\|_{L^\infty(M)} = 0, \tag{6.73}$$

and hence

$$\lim_{i\to\infty} \left\| u_i - u - \sum_{j=1}^l \omega_i^j \right\|_{H^1(M)} = 0, \tag{6.74}$$

where

$$\omega_i^j(\cdot) = \omega_j\left(\frac{\cdot - a_i^j}{\lambda_i^j}\right) - \omega_j(\infty).$$

Proof. Following the scheme by Brezis-Coron [18](see also [157] and [206]), we only need to consider the situation where there are two bubbles at the same point by two different scales and prove that there is no energy concentration and oscillation at the (neck) region between these two bubbles. More precisely, we assume that there exist $\lambda_i \to 0$ and $\mu_i \to 0$, with $\frac{\mu_i}{\lambda_i} \to \infty$, such that $u_i(x_1 + \lambda_i \cdot)$ converges to a nontrivial harmonic map ω_1 in $H^1_{\text{loc}}(\mathbb{R}^2, S^{L-1})$, and $u_i(x_1 + \mu_i \cdot)$ converges to another nontrivial harmonic map ω_2 in $H^1_{\text{loc}}(\mathbb{R}^2 \setminus \{0\}, S^{L-1})$. Moreover, for some sufficiently small constant ϵ_0 to be chosen later,

$$\int_{B_{r\mu_i} \setminus B_{R\lambda_i}} |\nabla u_i|^2 \leq \epsilon_0^2. \tag{6.75}$$

We need to show

Lemma 6.5.2 *Under the same notions as above, we have*

$$\lim_{i \to \infty} \int_{B_{\frac{r\mu_i}{2}} \setminus B_{2R\lambda_i}} |\nabla u_i|^2 = o(r, R^{-1}), \tag{6.76}$$

where $\lim_{r \to 0, R \to \infty} o(r, R^{-1}) = 0$.

Proof. Set $r_i = \frac{r\mu_i}{\lambda_i}$, we have $r_i \to \infty$. For simplicity, assume that $x_1 = 0$ and $B_{r\mu_i}$ is an Euclidean ball. Define $v_i(x) = u_i(\lambda_i x)$ for $x \in B_{r_i}$. Then v_i satisfies:

$$\tau(v_i) = \Delta v_i + |\nabla v_i|^2 v_i = \overline{h}_i \text{ in } B_{r_i}, \tag{6.77}$$

where $\overline{h}_i(x) = \lambda_i^2 h_i(\lambda_i x)$ for $x \in B_{r_i}$. By the conformal invariance of the Dirichlet energy in dimension two, we have

$$\int_{B_{r_i} \setminus B_R} |Dv_i|^2 \leq \epsilon_0^2. \tag{6.78}$$

For $1 \leq k, l \leq L$, consider the 1-forms $dv_i^k v_i^l - v_i^l dv_i^k$. Then (6.77) gives

$$d^* \left(dv_i^k v_i^l - v_i^l dv_i^k \right) = \overline{h}_i^k v_i^l - \overline{h}_i^l v_i^k \equiv H_i^{kl} \text{ in } B_{r_i} \setminus B_R, \tag{6.79}$$

hence

$$\Delta \left(dv_i^k v_i^l - v_i^l dv_i^k \right) = dH_i^{kl} + 2d^*(dv_i^k \wedge dv_i^l) \text{ in } B_{r_i} \setminus B_R. \tag{6.80}$$

Now, let $\overline{v}_i : \mathbb{R}^2 \to \mathbb{R}^L$ be an extension of v_i from $B_{r_i} \setminus B_R$ such that

$$\int_{\mathbb{R}^2} |\nabla \overline{v}_i|^2 \leq C \int_{B_{r_i} \setminus B_R} |\nabla v_i|^2 \leq C\epsilon_0^2. \tag{6.81}$$

Let $\overline{H}_i^{kl} : \mathbb{R}^2 \to \mathbb{R}^{L \times L}$ be an extension of H_i from B_{r_i} such that $\overline{H}_i^{kl} = 0$ outside B_{r_i}. Hence

$$\int_{\mathbb{R}^2} |\overline{H}_i|^p \leq C \int_{B_{r_i}} |H_i|^p \leq C\lambda_i^{2(p-1)} \int_{B_{\lambda_i \mu}} |h_i|^p. \tag{6.82}$$

By the Hodge decomposition theorem of Iwaniec-Martin [101], there are

$$\Psi_i \in H^1 \left(\mathbb{R}^2, \wedge^2(\mathbb{R}^{L \times L}) \right) \quad \text{and } F_i \in W^{2,p} \left(\mathbb{R}^2, \mathbb{R}^{L \times L} \right)$$

such that

$$\Delta\Psi_i^{kl} = d\overline{v}_i^k \wedge d\overline{v}_i^l, \tag{6.83}$$

and

$$\Delta F_i^{kl} = \overline{H}_i^{kl}. \tag{6.84}$$

Then we have

$$\Delta\left(dv_i^k v_i^l - v_i^k dv_i^l - dF_i^{kl} - 2d^*\Psi_i^{kl}\right) = 0 \text{ in } B_{r_i} \setminus B_R. \tag{6.85}$$

Therefore, if we define the 1-forms $G_i \in H^1(B_{r_i} \setminus B_R, \wedge(\mathbb{R}^{L\times L}))$ by

$$\begin{aligned} \Delta G_i^{kl} &= 0 \text{ in } B_{r_i} \setminus B_R, & (6.86)\\ i^*\left(G_i^{kl} - (dv_i^k v_i^l - v_i^k dv_i^l - dF_i^{kl} - 2d^*\Psi_i^{kl})\right) &= 0, & (6.87) \end{aligned}$$

where $i : \partial\left(B_{r_i} \setminus B_R\right) \to \mathbb{R}^2$ is the inclusion map and i^* is the pull-back map from one forms. Then for $1 \le k, l \le L$,

$$dv_i^k v_i^l - v_i^k dv_i^l - dF_i^{kl} - 2d^*\Psi_i^{kl} = G_i^{kl} \text{ in } B_{r_i} \setminus B_R. \tag{6.88}$$

For Ψ_i^{kl}, we observe that the right hand side of (6.83) is in the Hardy space $\mathcal{H}^1(\mathbb{R}^2)$ (see Chapter 3.2 or [35]) and also in $H^{-1}(\mathbb{R}^2)$ by [18]. Hence $\Psi_i^{kl} \in W^{2,1}(\mathbb{R}^2)$ and satisfies

$$\begin{aligned} \left\|\nabla^2\Psi_i^{kl}\right\|_{L^1(\mathbb{R}^2)} &\le \left\|d\overline{v}_i^k \wedge d\overline{v}_i^l\right\|_{\mathcal{H}^1(\mathbb{R}^2)} \le C\int_{\mathbb{R}^2} |\nabla\overline{v}_i|^2 \\ &\le C\int_{B_{r_i}\setminus B_R} |\nabla v_i|^2 \le C\epsilon_0^2, \end{aligned} \tag{6.89}$$

and

$$\int_{\mathbb{R}^2} \left|\nabla\Psi_i^{kl}\right|^2 \le C\int_{\mathbb{R}^2} |\nabla\overline{v}_i|^2 \le C\int_{B_{r_i}\setminus B_R} |\nabla v_i|^2 \le C\epsilon_0^2. \tag{6.90}$$

For F_i^{kl}, by $W^{2,p}$-estimate we have

$$\begin{aligned} \left\|\nabla^2 F_i^{kl}\right\|_{L^p(\mathbb{R}^2)} &\le C\left\|\overline{H}_i^{kl}\right\|_{L^p(\mathbb{R}^2)} \le C\left\|H_i^{kl}\right\|_{L^p(B_{r_i})} \\ &\le C\lambda_i^{2-\frac{2}{p}}\|h_i\|_{L^p(B_{\alpha\mu_i})} \le C\lambda_i^{2-\frac{2}{p}}. \end{aligned} \tag{6.91}$$

Hence, by Hölder inequality we have

$$\begin{aligned} \left\|\nabla^2 F_i^{kl}\right\|_{L^1(B_{r_i})} &\le \left\|\nabla^2 F_i^{kl}\right\|_{L^p(B_{r_i})} r_i^{2(1-\frac{1}{p})} \le \left\|\nabla^2 F_i^{kl}\right\|_{L^p(\mathbb{R}^2)} r_i^{2(1-\frac{1}{p})} \\ &\le C\|h_i\|_{L^p(B_{\alpha\mu_i})} \mu_i^{2(1-\frac{1}{p})} \le C\mu_i^{2(1-\frac{1}{p})}. \end{aligned} \tag{6.92}$$

This, combined with $W^{1,1}(\mathbb{R}^2) \subset L^{2,1}(\mathbb{R}^2)$, yields

$$\left\|\nabla F_i^{kl}\right\|_{L^{2,1}(B_{r_i})} \le C\|h_i\|_{L^p(B_{r\mu_i})} \mu_i^{2(1-\frac{1}{p})}. \tag{6.93}$$

Moreover, using the $L^{2,\infty}$-estimate for ∇F_i we have

$$\begin{aligned}\left\|\nabla F_i^{kl}\right\|_{L^{2,\infty}(\mathbb{R}^2)} &\leq C\left\|\overline{H}_i^{kl}\right\|_{L^1(\mathbb{R}^2)} \leq C\left\|H_i\right\|_{L^1(B_{r_i})}\\ &\leq C\left\|h_i\right\|_{L^1(B_{r\mu_i})} \leq C\left\|h_i\right\|_{L^p(B_{r\mu_i})}\mu_i^{2(1-\frac{1}{p})}\\ &\leq C\mu_i^{2(1-\frac{1}{p})}. \end{aligned} \tag{6.94}$$

Using the duality between $L^{2,1}$ and $L^{2,\infty}$, we have

$$\begin{aligned}\left\|\nabla F_i^{kl}\right\|_{L^2(B_{r_i})} &\leq \left\|\nabla F_i^{kl}\right\|_{L^{2,1}(B_{r_i})}^{\frac{1}{2}}\left\|\nabla F_i^{kl}\right\|_{L^{2,\infty}(B_{r_i})}^{\frac{1}{2}}\\ &\leq C\left\|h_i\right\|_{L^p(B_{r\mu_i})}\mu_i^{2(1-\frac{1}{p})}. \end{aligned} \tag{6.95}$$

For G_i^{kl}, we can choose suitable $r>0$ and $R>0$ so that

$$R^{\frac{1}{2}}\left\|G_i^{kl}\right\|_{L^2(\partial B_R)} \leq C,\ r_i^{\frac{1}{2}}\left\|G_i^{kl}\right\|_{L^2(\partial B_{r_i})} \leq C.$$

This implies (since G_i is a harmonic 1-form)

$$\max\left\{\left\|G_i\right\|_{L^{2,1}(B_{\frac{r_i}{2}}\setminus B_{2R})},\ \left\|G_i\right\|_{L^{2,\infty}(B_{\frac{r_i}{2}}\setminus B_{2R})}\right\} \leq CR^{-\frac{1}{2}}. \tag{6.96}$$

Substituting these estimates into (6.88), we have that $\left(dv_i^k v_i^l - v_i^l dv_i^k - 2d^*\Psi_i^{kl}\right) \in L^{2,1}\cap L^{2,\infty}(B_{\frac{r_i}{2}}\setminus B_{2R})$ and

$$\left\|(dv_i^k v_i^l - v_i^k dv_i^l - 2d^*\Psi_i^{kl})\right\|_{L^{2,1}(B_{\frac{r_i}{2}}\setminus B_{2R})} \leq C\left(R^{-\frac{1}{2}} + \mu_i^{2(1-\frac{1}{p})}\right), \tag{6.97}$$

$$\left\|(dv_i^k v_i^l - v_i^k dv_i^l - 2d^*\Psi_i^{kl})\right\|_{L^{2,\infty}(B_{\frac{r_i}{2}}\setminus B_{2R})} \leq C\left(R^{-\frac{1}{2}} + \mu_i^{2(1-\frac{1}{p})}\right). \tag{6.98}$$

Hence

$$\left\|dv_i^k v_i^l - v_i^k dv_i^l - 2d^*\Psi_i^{kl})\right\|_{L^2(B_{\frac{r_i}{2}}\setminus B_{2R})} \leq C\left(R^{-\frac{1}{2}} + \mu_i^{2(1-\frac{1}{p})}\right). \tag{6.99}$$

Therefore we have

$$\begin{aligned}\left\|dv_i^k v_i^l - v_i^k dv_i^l\right\|_{L^2(B_{\frac{r_i}{2}}\setminus B_{2R})} &\leq 2\left\|\nabla\Psi_i^{kl}\right\|_{L^2(\mathbb{R}^2)} + CR^{-\frac{1}{2}} + C\mu_i^{2(1-\frac{1}{p})}\\ &\leq C\int_{B_{\frac{r_i}{2}}\setminus B_R}|\nabla v_i|^2 + CR^{-\frac{1}{2}} + C\mu_i^{2(1-\frac{1}{p})}\\ &\leq C\epsilon_0\left\|\nabla v_i\right\|_{L^2(B_{\frac{r_i}{2}}\setminus B_{2R})} + C\int_{B_{r_i}\setminus B_{\frac{r_i}{2}}}|\nabla v_i|^2\\ &\quad + C\int_{B_{2R}\setminus B_R}|\nabla v_i|^2 + CR^{-\frac{1}{2}} + C\mu_i^{2(1-\frac{1}{p})}. \end{aligned}$$

Since

$$\sum_{k,l=1}^{L} \left| dv_i^k v_i^l - v_i^k dv_i^l \right|^2 = 2 \left| \nabla v_i \right|^2,$$

we have by choosing ϵ_0 sufficiently small and summing the left hand side of the above inequality over $1 \leq k, l \leq L$,

$$\begin{aligned} \|\nabla v_i\|_{L^2(B_{\frac{r_i}{2}} \setminus B_{2R})} &\leq C \left(\int_{B_{r_i} \setminus B_{\frac{r_i}{2}}} |\nabla v_i|^2 + \int_{B_{2R} \setminus B_R} |\nabla v_i|^2 \right) \\ &\quad + CR^{-\frac{1}{2}} + C\mu_i^{2(1-\frac{1}{p})}. \end{aligned} \tag{6.100}$$

Since

$$\int_{B_{2R} \setminus B_R} |\nabla v_i|^2 = \int_{B_{2R} \setminus B_R} |\nabla \omega_1|^2 + o(i^{-1}),$$

and

$$\int_{B_{r_i} \setminus B_{\frac{r_i}{2}}} |\nabla v_i|^2 \leq \int_{B_r \setminus B_{\frac{r}{2}}} |\nabla \omega_2|^2 + o(i^{-1})$$

where $\lim_{i \to \infty} o(i^{-1}) = 0$, it is clear that if we choose R sufficiently large and r sufficiently small, then both terms in the right hand sides of the above two inequalities can be arbitrarily small. This completes the proof of Lemma 6.5.2. □

The oscillation convergence in Theorem 6.5.1 also follows from Lemma 6.5.2. In fact, it follows from the proof of Lemma 6.5.2 that

$$\left\| \nabla^2 v_i \right\|_{L^1(B_{\frac{r_i}{2}} \setminus B_{2R})} \leq C \int_{B_{r_i} \setminus B_R} |\nabla v_i|^2 + C \|h_i\|_{L^p} \mu_i^{2(1-\frac{1}{p})} \to 0, \tag{6.101}$$

as $i \to \infty$ and $R \to \infty$. Let $\tilde{v}_i$ be an extension of v_i to $\mathbb{R}^2$ such that it is compactly supported and

$$\left\| \nabla^2 \tilde{v}_i \right\|_{L^1(\mathbb{R}^2)} \leq C \left\| \nabla^2 v_n \right\|_{L^1(B_{\frac{r_i}{2}} \setminus B_{2R})}.$$

Since

$$\begin{aligned} |\tilde{v}_i|(x) &= \left| \int_{\mathbb{R}^2} \log|x-y| \Delta \tilde{v}_i(y) dy \right| \\ &= \left| \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \nabla \tilde{v}(y) dy \right| \\ &\leq \left\| \frac{y}{|y|^2} \right\|_{L^{2,\infty}(\mathbb{R}^2)} \cdot \|\nabla \tilde{v}_i\|_{L^{2,1}(\mathbb{R}^2)} \\ &\leq C \left\| \nabla^2 \tilde{v}_i \right\|_{L^1(\mathbb{R}^2)} \\ &\leq C \left\| \nabla^2 v_i \right\|_{L^1(B_{\frac{r_i}{2}} \setminus B_{2R})}, \end{aligned}$$

we have

$$\lim_{R \to \infty} \lim_{i \to \infty} \max_{x \in B_{\frac{r_i}{2}} \setminus B_{2R}} |v_i|(x) = 0.$$

This implies that there is no neck formation between any two bubbles. Hence the proof of Theorem 6.73 is complete. □

To conclude this section, we present an essentially optimal condition on th tension fields, that is satisfied if $\tau(u_i)$ is bounded in $L^p(M)$ for any $p > 1$. For an bounded domain $\Omega \subset \mathbb{R}^2$, denote by $\mathcal{H}^1(\Omega)$ the local Hardy space on Ω defined by Semmes [165]. For a Riemannian surface M without boundary, we define $\mathcal{H}^1(M)$ by using coordinate charts.

Proposition 6.5.3 *The energy identity part of Theorem 6.5.1 remains to hold if (a) the tension field $\tau(u_i)$ is uniformly bounded in $\mathcal{H}^1(M)$, and (b) $\tau(u_i)$ is equi-integrable, i.e. for any $\epsilon > 0$ there is $\delta > 0$ such that for any $E \subset M$, with $|E| \le \delta$, $\int_E |\tau(u_i)|(x)dx \le \epsilon$ for any $i \ge 1$.*

The oscillation convergence part of Theorem 6.5.1 remains to hold if $\tau(u_i)$ is equi-integrable in $\mathcal{H}^1(M)$ in the sense that for any $\epsilon > 0$ there is a $\delta > 0$ such that for any open set $E \subset M$, with $|E| \le \delta$, $\|\tau(u_i)\|_{\mathcal{H}^1(E)} \le \epsilon$ for any $i \ge 1$.

Proof. It is the same as that of Lemma 6.5.2, except that we estimate the $L^{2,1}$ norm of ∇F_i^{kl} in a different way. To do it, let $\eta \in C_0^1(\mathbb{R}^2, \mathbb{R}_+)$ be such that $\eta = 1$ in B_{r_i}, and

$$c_i^{kl} = \frac{\int_{\mathbb{R}^2} \eta H_i^{kl}}{\int_{\mathbb{R}^2} \eta}.$$

Then by a lemma of [165] we have that $\eta\left(H_i^{kl} - c_i^{kl}\right) \in \mathcal{H}^1(\mathbb{R}^2)$ and

$$\begin{aligned} \left\|\eta\left(H_i^{kl} - c_i^{kl}\right)\right\|_{\mathcal{H}^1(\mathbb{R}^2)} &\le C\left\|\overline{h}_i\right\|_{\mathcal{H}^1(B_{r_i})} \\ &\le C\left\|h_i\right\|_{\mathcal{H}^1(M)} \le C < \infty. \end{aligned} \tag{6.102}$$

Now let $F_{i,1}^{kl}$ and $F_{i,2}^{kl}$ be 1-forms on $\mathbb{R}^2$ such that

$$\Delta F_{i,1}^{kl} = \eta\left(H_i^{kl} - c_i^{kl}\right) \tag{6.103}$$

$$\Delta F_{i,2}^{kl} = c_i^{kl}\eta. \tag{6.104}$$

This gives

$$F_i^{kl} = F_{i,1}^{kl} + F_{i,2}^{kl} + L_i^{kl} \text{ in } B_{r_i}, \tag{6.105}$$

where L_i^{kl} is a harmonic 1-form on B_{r_i} so that the $L^{2,1}$ norm of ∇L_i^{kl} can be estimated in the same way as that of G_i^{kl}.

For $F_{i,1}^{kl}$ and $F_{i,2}^{kl}$, we have

$$\left\|\nabla^2 F_{i,1}^{kl}\right\|_{L^1(\mathbb{R}^2)} \le C\left\|\eta\left(H_i^{kl} - c_i^{kl}\right)\right\|_{\mathcal{H}^1(\mathbb{R}^2)} \le C,$$

and

$$\begin{aligned} \left\|\nabla^2 F_{i,2}^{kl}\right\|_{L^1(B_{r_i})} &\le Cr_i\left\|\nabla^2 F_{i,2}^{kl}\right\|_{L^2(B_{r_i})} \\ &\le Cr_i\left\|c_i^{kl}\eta\right\|_{L^2(\mathbb{R}^2)} = Cr_i^2|c_i^{kl}| \\ &\le C\left\|\overline{h}_i\right\|_{L^1(B_{r_i})} \\ &\le C\left\|h_i\right\|_{L^1(B_\delta)} \le C. \end{aligned}$$

From these two inequalities, we obtain a upper bound of $\left\|\nabla^2 F_i^{kl}\right\|_{L^1(B_{r_i})}$. The smallness of $\left\|\nabla F_i^{kl}\right\|_{L^{2,\infty}(B_{r_i})}$ follows from the equi-integrability condition of $\tau(u_i)$. In fact, for any $\epsilon > 0$, we can choose $\delta > 0$ sufficiently small so that

$$\left\|\eta H_i^{kl}\right\|_{L^1(\mathbb{R}^2)} \leq \left\|\overline{h}_i\right\|_{L^1(B_{r_i})} \leq \|h_i\|_{L^1(B_\delta)} \leq \epsilon.$$

If, in additions, h_i is equi-integrable in $\mathcal{H}^1(M)$, then the above argument implies that $\left\|\nabla F_i^{kl}\right\|_{L^{2,1}(B_{r_i})} = o(\delta)$ so that the oscillation convergence follows as well. □

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Chapter 7

Partially smooth heat flows

In this chapter, we will present the important work by Chen-Struwe [33] on the existence of global weak solutions to the heat flow of harmonic maps in higher dimensions, which is smooth away from a singular set of small size. There are two ingredients in the construction of [33]. One is the parabolic energy monotonicity inequality first discovered by Struwe [195] for smooth heat flows of harmonic maps. The other is the Ginzburg-Landau type approximation scheme for heat flow of harmonic maps. We remark that monotonicity inequalities similar to [195] have also played important roles in other geometric flows such as mean curvature flows (cf. Huisken [76]). This chapter is organized as follows. In §7.1, we derive the monotonicity formula by [195]. In §7.2, we show the existence of global smooth heat flows for small initial data by [195]. In §7.3, we present examples on finite time singularity of the heat flow of harmonic maps in higher dimensions by [39] and [28]. In §7.4, we present examples on non uniqueness of the heat flow of harmonic maps by [38]. In §7.5 and 7.6, we present the construction of partially smooth heat flows by [27] and [33].

7.1 Monotonicity formula and a priori estimates

This section is devoted to the monotonicity formula for smooth heat flows of harmonic maps by Struwe [195]. Main references for the presentation in this section are from [195] and [196]. First we briefly recall the definition of the parabolic Hausdorff measure on $\mathbb{R}^n \times \mathbb{R}$. The parabolic metric δ on $\mathbb{R}^n \times \mathbb{R}$ is given by

$$\delta\left((x,t),(y,s)\right) = \max\left\{|x-y|, \sqrt{|t-s|}\right\},\ (x,t),(y,s) \in \mathbb{R}^n \times \mathbb{R}.$$

For any $0 \le \alpha \le n+2$ and any set $E \subset \mathbb{R}^n \times \mathbb{R}$, α-dimensional parabolic Hausdorff measure of E is defined by

$$\begin{aligned}\mathcal{P}^\alpha(E) &= \lim_{\delta\to 0}\mathcal{P}^\alpha_\delta(E)\\ &= \lim_{\delta\to 0}\inf\left\{\sum_i r_i^\alpha \mid E \subset \bigcup_i Q_{r_i}(z_i),\ z_i \in E,\ r_i \le \delta\right\}\end{aligned}$$

where $Q_r(z_0) = \{z \in \mathbb{R}^n \times \mathbb{R} \mid |x-x_0| < r, |t-t_0| < r^2\}$.

For simplicity we assume $M = \mathbb{R}^n$ and consider a solution $u \in C^\infty(\mathbb{R}^n \times (0,+\infty), N)$ of (5.3) that satisfies the energy bound:

$$E(u(t)) \leq E_0 < +\infty, \ \forall t \in (0,+\infty). \tag{7.1}$$

For $z_0 = (x_0, t_0) \in \mathbb{R}^n \times (0,+\infty)$, let

$$G_{z_0}(x,t) = \frac{1}{\left(\sqrt{4\pi|t-t_0|}\right)^n} \exp\left(-\frac{|x-x_0|^2}{4|t-t_0|}\right), \ x \in \mathbb{R}^n, \ t < t_0,$$

be the backward heat kernel on $\mathbb{R}^n$, and for $R > 0$ denote

$$S_R(t_0) = \left\{(x,t) \mid x \in \mathbb{R}^n, \ t = t_0 - R^2\right\},$$

$$T_R(t_0) = \left\{(x,t) \in \mathbb{R}^n \times (0,+\infty) \mid t_0 - 4R^2 \leq t \leq t_0 - R^2\right\}.$$

Using G_{z_0} as a weight function, we define two energy quantities:

$$\Phi_{z_0}(\rho) := \Phi_{z_0}(\rho; u) = \frac{1}{2}\rho^2 \int_{\mathbb{R}^n \times \{t_0 - \rho^2\}} |\nabla u|^2 G_{z_0} \, dx, \ 0 < \rho < \sqrt{t_0}, \tag{7.2}$$

and

$$\Psi_{z_0}(R) := \Psi_{z_0}(R; u) = \int_{T_R(t_0)} |\nabla u|^2 G_{z_0} \, dxdt, \ 0 < R < \frac{\sqrt{t_0}}{2}. \tag{7.3}$$

Then we have

Theorem 7.1.1 *For any $z_0 = (x_0, t_0) \in \mathbb{R}^n \times (0,+\infty)$ and $0 < \rho \leq r < \sqrt{t_0}$, it holds*

$$\Phi_{z_0}(\rho) \leq \Phi_{z_0}(r). \tag{7.4}$$

Proof. For simplicity, assume $z_0 = (0,0)$ and $u \in C^\infty(\mathbb{R}^n \times (-\infty, 0), N)$. Denote $G = G_{(0,0)}$ and $\Phi(\rho) = \Phi_{(0,0)}(\rho)$. First observe that for any $R > 0$,

$$u_R(x,t) = u(Rx, R^2 t), \ (x,t) \in \mathbb{R}^n \times (-\infty, 0)$$

also solves (5.3). Note also that $G(Rx, R^2 t) = R^{-n} G(x,t)$. Hence we have $\Phi(\rho) = \Phi(1; u_\rho)$. Direct calculations imply

$$\begin{aligned}
\frac{d}{d\rho}\Phi(\rho) &= \frac{d}{d\rho}\Phi(1; u_\rho) \\
&= \int_{\mathbb{R}^n \times \{-1\}} \nabla u_\rho \cdot \nabla \left(\frac{du_\rho}{d\rho}\right) G \, dx \\
&= -\int_{\mathbb{R}^n \times \{-1\}} \nabla \cdot (G \nabla u_\rho) \cdot \left(\frac{du_\rho}{d\rho}\right) dx
\end{aligned}$$

Since

$$\frac{du_\rho}{d\rho} = (x \cdot \nabla u + 2\rho t u_t)\left(\rho x, \rho^2 t\right), \ \nabla G = \frac{x}{2t} G,$$

by using (5.3) we obtain

$$\begin{aligned}\frac{d}{d\rho}\Phi(\rho) &= \frac{1}{2\rho}\int_{\mathbb{R}^n\times\{-1\}} G\left|\rho x\cdot\nabla u - 2\rho^2\partial_t u\right|^2(\rho x,\rho^2 t)\,dx\\ &= \frac{1}{2\rho}\int_{\mathbb{R}^n\times\{-\rho^2\}} \left|x\cdot\nabla u + 2t\partial_t u\right|^2 G\,dx \geq 0. \qquad (7.5)\end{aligned}$$

This implies (7.4).

Remark 7.1.2 Integrating (7.5) over $0<\rho_1<\rho_2<\sqrt{t_0}$, we actually get

$$\Phi_{z_0}(\rho_2)-\Phi_{z_0}(\rho_1) = \int_{\rho_1}^{\rho_2}\frac{1}{2\rho}\int_{\mathbb{R}^n\times\{t_0-\rho^2\}}\left|x\cdot\nabla u + 2t\partial_t u\right|^2 G\,dx\,d\rho. \qquad (7.6)$$

(7.6) plays a very important role in the blow up analysis of (5.3) in Chapters 8 and 9 below.

Next we have

Theorem 7.1.3 *Suppose that $u\in C^\infty(\mathbb{R}^n\times(0,+\infty),N)$ solves (5.3). Then for any $z_0\in\mathbb{R}^n\times(0,+\infty)$, we have*

$$\Psi_{z_0}(r)\leq\Psi_{z_0}(R),\quad 0<r\leq R<\sqrt{t_0}. \qquad (7.7)$$

Proof. Note that

$$\begin{aligned}\Psi_{z_0}(R) &= \int_{t_0-4R^2}^{t_0-R^2}\int_{\mathbb{R}^n}|\nabla u|^2 G_{z_0}\,dxdt\\ &= \int_R^{2R}\left(\rho^2\int_{\mathbb{R}^n\times\{t_0-\rho^2\}}|\nabla u|^2 G_{z_0}\,dx\right)\frac{d\rho}{\rho}\\ &= \int_R^{2R}\Phi_{z_0}(\rho)\frac{d\rho}{\rho}.\end{aligned}$$

Hence (7.7) follows from (7.4). □

As an immediate consequence of (7.4), we can derive an apriori gradient estimates for smooth solutions u of (5.3), under the smallness of $\Phi(\rho)$.

Proposition 7.1.4 *There exists $\epsilon_0>0$ depending only on n,N such that for any solution $u\in C^\infty(\mathbb{R}^n\times(-\infty,0),N)$ of (5.3), if $\Phi(R)\leq\epsilon_0^2$ for some $R>0$, then*

$$\sup_{P_{\delta R}}\left(R^2|\partial_t u|+R|\nabla u|\right)\leq C \qquad (7.8)$$

for some positive constants δ and C depending only on ϵ_0,n,N,E_0.

Proof. By scaling, we may assume $R=1$. For $\delta>0$, let $\rho\in[0,\delta]$, $z_0=(x_0,t_0)\in P_\rho$ be such that

$$(\delta-\rho)^2\sup_{P_\rho}e(u) = \max_{0\leq\sigma\leq\rho}\left\{(\delta-\sigma)^2\sup_{P_\sigma}e(u)\right\},\ e(u)(z_0)=\sup_{P_\rho}e(u)=e_0. \qquad (7.9)$$

We may assume $e_0 \geq \frac{4}{(\delta-\rho)^2}$. For, otherwise, we are done. Now we rescale u and define $v(x,t) = u(x_0 + \frac{x}{\sqrt{e_0}}, t_0 + \frac{t}{e_0}) \in C^\infty(P, N)$. Then we have $e(v)(0,0) = 1$ and

$$\begin{aligned}\sup_P e(v) &= e_0^{-1} \sup_{P_{e_0^{-\frac{1}{2}}}(z_0)} e(u) \leq e_0^{-1} \sup_{P_{\frac{\delta+\rho}{2}}} e(u) \\ &\leq 4e_0^{-1} \sup_{P_\rho} e(u) = 4. \end{aligned} \tag{7.10}$$

This, combined with (5.3.3), implies

$$\left(\frac{\partial}{\partial t} - \Delta\right) e(v) \leq Ce(v) \quad \text{in} \quad P. \tag{7.11}$$

Hence, by the Harnack inequality we have

$$1 = e(v)(0,0) \leq C \int_P e(v)\, dxdt = Ce_0^{\frac{n}{2}} \int_{P_{e_0^{-\frac{1}{2}}}(z_0)} e(u)\, dxdt. \tag{7.12}$$

Let $z_1 = (x_1, t_1) = z_0 + (0, e_0^{-1})$. Then by (7.4) we have

$$\begin{aligned} 1 &\leq C \int_{P_{e_0^{-\frac{1}{2}}}(z_0)} e(u) G_{z_1}\, dxdt \leq C \int_{e_0^{-\frac{1}{2}}}^{2e_0^{-\frac{1}{2}}} \Phi(\rho) \frac{d\rho}{\rho} \\ &\leq C \int_{\mathbb{R}^n \times \{-1\}} e(u) G_{z_1}\, dx. \end{aligned}$$

Since $|x_1| \leq \delta$ and $|t_1| \leq \delta^2$, we can estimate, at $t = 1$,

$$\begin{aligned} |G_{z_1} - G| &\leq \left(\left| 1 - \frac{1}{(\sqrt{|1+t_1|})^n} \right| + \left| \exp(-\frac{|x|^2}{4}) - \exp(-\frac{-|x-x_1|^2}{4|1+t_1|}) \right| \right) \\ &\leq C\delta, \end{aligned}$$

and hence we obtain

$$1 \leq C\delta \int_{\mathbb{R}^n \times \{-1\}} e(u)\, dx + C \int_{\mathbb{R}^n \times \{-1\}} e(u) G\, dx \leq C\left(\delta E_0 + \Phi(1)\right). \tag{7.13}$$

By choosing both δ and ϵ_0 sufficiently small, the above inequality leads to a contradiction. Thus we must have $e_0(\delta - \rho)^2 \leq 4$ and hence

$$\sup_{P_{\frac{\delta}{2}}} e(u) \leq 16\delta^{-2}.$$

To obtain a pointwise bound on $\partial_t u$, we first compute

$$\begin{aligned} (\partial_t - \Delta) |\partial_t u|^2 &= 2\partial_t u \cdot \partial_t (\partial_t u - \Delta u) - 2 |\nabla \partial_t u|^2 \\ &= 2\partial_t u \cdot \partial_t (A(u)(\nabla u, \nabla u)) - 2 |\nabla \partial_t u|^2 \\ &= 2\partial_t u \cdot \nabla_u A(u) (\nabla u, \nabla u) \partial_t u \\ &\quad +4\partial_t u \cdot A(u) (\nabla \partial_t u, \nabla u) - 2 |\nabla \partial_t u|^2 \\ &\leq C |\nabla u|^2 |\partial_t u|^2 + C |\nabla \partial_t u| |\partial_t u| |\nabla u| - 2 |\nabla \partial_t u|^2 \\ &\leq C |\nabla u|^2 |\partial_t u|^2 \leq C |\partial_t u|^2. \end{aligned}$$

Hence by the Harnack inequality we have

$$\max_{P_{\frac{R}{2}}(z)} |\partial_t u|^2 \le CR^{-(n+2)} \int_{P_R(z)} |\partial_t u|^2 \le CR^{-(n+4)} \int_{P_{2R}(z)} |\nabla u|^2,$$

where C depends on M, N and $\|\nabla u\|_{L^\infty(P_R(z))}$. The proof is complete. □

7.2 Global smooth solutions and weak compactness

For $n \ge 3$, it is an important question to find sufficient conditions other than the curvature condition on N to guarantee the existence of globally smooth solutions to the heat flow of harmonic maps. As an immediate corollary of Lemma 7.1.1 and Proposition 7.1.4, Struwe [195] has proved

Theorem 7.2.1 *There exists $\epsilon_1 > 0$ depending only n, N and $\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}$ such that for any initial data $u_0 \in W^{1,2}(\mathbb{R}^n, N) \cap \mathrm{Lip}(\mathbb{R}^n, N)$ if $E(u_0) \le \epsilon_1$, then there exists a unique smooth solution $u \in C^\infty(\mathbb{R}^n \times [0, +\infty), N)$ to (5.3), with $u|_{t=0} = u_0$. Moreover,*

$$\lim_{t\uparrow+\infty} u(t) = p \text{ for some point } p \in N.$$

Proof. Note that for $n = 2$, Theorem 7.2.1 follows from Theorem 6.2.1. Hence we assume $n \ge 3$. Observe that for any such a u_0, we have the Morrey space type estimate:

$$\begin{aligned}
&\|\nabla u_0\|_{M^{2,n-2}(\mathbb{R}^n)} \\
\le\ & \inf_{R_0>0} \left\{ \sup_{x_0\in\mathbb{R}^n, 0<R\le R_0} R^{2-n} \int_{B_R(x_0)} |\nabla u_0|^2, \sup_{x_0\in\mathbb{R}^n, R>R_0} R^{2-n} \int_{B_R(x_0)} |\nabla u_0|^2 \right\} \\
\le\ & \inf_{R_0>0} \left\{ \|\nabla u_0\|^2_{L^\infty(\mathbb{R}^n)} R_0^2 + R_0^{2-n} E_0 \right\} \\
\le\ & \inf_{R_0>0} \left(R_0^2 \|\nabla u_0\|^2_{L^\infty(\mathbb{R}^n)} + R_0^{2-n} \epsilon_1 \right) \\
\le\ & \epsilon_0^2
\end{aligned} \tag{7.14}$$

provided that we choose

$$R_0 = \frac{\epsilon_0}{2\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)}}, \quad \epsilon_1 \le \frac{\epsilon_0^n}{4^n \|\nabla u_0\|^{n-2}_{L^\infty(\mathbb{R}^n)}}.$$

To proceed, we need another lemma.

Lemma 7.2.2 *There exist a sequence $\{u_0^k\} \subset C^\infty(\mathbb{R}^n, N)$ such that*

$$E(u_0^k) \le CE(u_0), \quad \|\nabla u_0^k\|_{L^\infty(\mathbb{R}^n)} \le C\|\nabla u_0\|_{L^\infty(\mathbb{R}^n)} \text{ for all } k,$$

and $\nabla u_0^k \to \nabla u_0$ in $L^2(\mathbb{R}^n)$.

Proof. Let $\delta_0 > 0$ be so small that the nearest point projection map $\Pi_N : N_{\delta_0} \to N$ is smooth. For $0 < \epsilon < 1$, let $u_0^\epsilon = u_0 * \phi_\epsilon : \mathbb{R}^n \to \mathbb{R}^L$ be a standard mollification of u. It is well-known that $u_0^\epsilon \in C^\infty(\mathbb{R}^n, \mathbb{R}^L)$ and $u_\epsilon \to u$ strongly in $W^{1,2}_{\text{loc}}(\mathbb{R}^n, \mathbb{R}^L)$ as $\epsilon \to 0$. Moreover, by a modified Poincaré inequality and (7.14), we have

$$\begin{aligned} \text{dist}\left(u_0^\epsilon(x), N\right)^2 &\leq CR^{-n} \int_{B_R(x)} \left|u_0(y) - u_0^\epsilon(x)\right|^2 \, dy \\ &\leq CR^{2-n} \int_{B_R(x)} \left|\nabla u_0\right|^2 \, dy \leq C\epsilon_0^2, \quad \forall x \in \mathbb{R}^n \end{aligned}$$

Hence $v_0^\epsilon = \Pi_N(u_0^\epsilon) \in C^\infty(\mathbb{R}^n, N)$. Moreover, it is easy to see that

$$\|\nabla v_0^\epsilon\|_{L^2(\mathbb{R}^n)} \leq C \|\nabla u_0^\epsilon\|_{L^2(\mathbb{R}^n)} \leq C \|\nabla u_0\|_{L^2(\mathbb{R}^n)} \leq C\epsilon_1,$$

$$\|\nabla v_0^\epsilon\|_{L^\infty(\mathbb{R}^n)} \leq C \|\nabla u_0^\epsilon\|_{L^\infty(\mathbb{R}^n)} \leq C \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)},$$

and $v_0^\epsilon \to u_0$ strongly in $W^{1,2}_{\text{loc}}(\mathbb{R}^n, N)$. This proves the lemma. □

By considering (5.3) with initial data u_0 replaced by v_0^ϵ, we may further assume $u_0 \in C^\infty(\mathbb{R}^n, N)$. We now want to show that for $\epsilon > 0$ sufficiently small,

$$\sup_{x_0 \in \mathbb{R}^n, R>0} R^2 \int_{\mathbb{R}^n} \left|\nabla u_0\right|^2 G_{(x_0, R^2)} \, dx \leq \epsilon_0^2. \tag{7.15}$$

In fact

$$\begin{aligned} R^2 \int_{\mathbb{R}^n} \left|\nabla u_0\right|^2 G_{(x_0,R^2)} \, dx &\leq CR^{2-n} \int_{B_{2R\ln R}(x_0)} \left|\nabla u_0\right|^2 \, dx \\ &\quad + CR^{2-n} \exp\left(-|\ln R|^2\right) E_0 \\ &\leq CR^2 |\ln R|^n \left\|\nabla u_0\right\|^2_{L^\infty(\mathbb{R}^n)} \\ &\quad + CR^{2-n} \exp\left(-|\ln R|^2\right) E_0 \leq \epsilon_0^2 \end{aligned}$$

provided that we choose $R \leq R_1(E_0, \|\nabla u_0\|_{L^\infty(\mathbb{R}^n)})$. For $R \geq R_1$, it is easy to see

$$R^2 \int_{\mathbb{R}^n} |\nabla u_0|^2 G_{(x_0,R^2)} \, dx \leq R_1^{2-n} E_0 \leq \epsilon_0^2$$

if E_0 is chosen to be sufficiently small.

Let $0 < T \leq +\infty$ be the maximal time interval such that there is a unique smooth solution $u \in C^\infty(\mathbb{R}^n \times [0, T), N)$ of (5.3) with the inital data u_0. Applying (7.4), we have

$$\Phi_{(x_0,T)}(r; u) \leq \epsilon_0^2, \quad \forall x_0 \in \mathbb{R}^n, \quad 0 < r < \sqrt{T}. \tag{7.16}$$

Hence Proposition 7.1.4 implies that if $T < +\infty$, then

$$\left|\nabla u\right|(x, T) \leq \frac{C}{\sqrt{T}}. \tag{7.17}$$

This contradicts the definition of T. Hence $T = +\infty$. In this case, there is $C > 0$ such that

$$\left|\nabla u\right|(x, t) \leq \frac{C}{\sqrt{t}}, \ \forall x \in \mathbb{R}^n \ \text{ and } t \geq 1.$$

This implies that $u(t) \to p$ for some $p \in N$, as $t \uparrow +\infty$. □

As in [195, 196], we can prove the following weak compactness theorem.

Theorem 7.2.3 *Suppose that* $\{u_k\} \subset C^\infty(\mathbb{R}^n \times (-1,0), N)$ *is a sequence of solutions of* (5.3) *with* $E(u_k(t)) \le E_0 < +\infty$ *uniformly in* t, *for* $k \in \mathbb{N}$. *Moreover, suppose* $u_k(-1) \to u$ *in* $H^1_{loc}(\mathbb{R}^n, N)$ *as* $k \to \infty$ *and*

$$u_k \to u \quad in \quad L^2_{loc}(\mathbb{R}^n \times [-1,0]),$$

$$\partial_t u_k \to \partial_t u \quad weaky\, in \quad L^2_{loc}(\mathbb{R}^n \times [-1,0]),$$

$$\nabla u_k \to \nabla u \quad weakly \quad in \quad L^2_{loc}(\mathbb{R}^n \times [-1,0])$$

Then u *is a weak solution of* (5.3) *and there exists a closed set* Σ *of locally finite* n*-dimensional parabolic Hausdorff measure such that* $u \in C^\infty(\mathbb{R}^n \times (-1,0) \setminus \Sigma, N)$. *Moreover for any* $0 < R < 1$, *we have*

$$\Phi(R) + \int_{-1}^{-R^2} \int_{\mathbb{R}^n} \frac{|x \cdot \nabla u + 2tu_t|^2}{2|t|} G\, dxdt \le \Phi(-1). \tag{7.18}$$

Proof. For $z_0 = (x_0, t_0) \in \mathbb{R}^n \times [-1,0]$, denote

$$\Phi^k_{z_0}(r) = \begin{cases} \Phi_{z_0}(r; u_k), & \text{if } 0 < r \le \sqrt{1+t_0}; \\ \Phi_{z_0}(\sqrt{1+t_0}; u_k) & \text{otherwise.} \end{cases}.$$

Define the concentration set

$$\Sigma = \bigcap_{r>0} \left\{ z \in \mathbb{R}^n \times [-1,0] \mid \varliminf_{k\to\infty} \Phi^k_z(r) \ge \epsilon_0^2 \right\},$$

where $\epsilon_0 >$ is given by Proposition 7.1.4. Then Σ is a closed subset of $\mathbb{R}^n \times [-1,0]$. To see this, let $\bar{z} \in \overline{\Sigma}$ and $z_l \in \Sigma$, with $z_l \to \bar{z}$. By the definition of Σ, we have

$$\varliminf_{l\to\infty} \varliminf_{k\to\infty} \left(\frac{r^2}{2} \int_{\mathbb{R}^n \times \{\bar{t}-r^2\}} |\nabla u_k|^2 G_{z_l}\, dx \right) \ge \epsilon_0^2, \ \forall r > 0. \tag{7.19}$$

Since $G_{z_l} \to G_{\bar{z}}$ uniformly away from $\bar{z}$ and

$$E(u_k(t)) \le E_0 := E\left(u(-1)\right) < +\infty \text{ uniformly for } k \text{ and } t,$$

for any fixed $r > 0$ we can interchange the limiting process to get

$$\varliminf_{k\to\infty} \Phi^k_{\bar{z}}(r) \ge \epsilon_0^2, \quad \forall r > 0,$$

this implies $\bar{z} \in \Sigma$.

For any $z_0 \notin \Sigma$, there exist $r_0 > 0$ and a subsequence of $\{u_k\}$, denoted as itself, such that

$$\Phi^k_{z_0}(r_0) \le \epsilon_0^2.$$

Proposition 7.1.4 implies that

$$\sup_{P_{\delta r_0}(z_0)} |\nabla u_k| \leq C r_0^{-1}, \quad \forall k,$$

for some positive constants δ and C only depending on n, N, E_0. By the higher order regularity, similar bounds hold for derivatives of any order. Thus, by sending $k \to \infty$, we conclude that $u_k \to u$ in $C^m\left(P_{\frac{r_0}{2}}(z_0), N\right)$ for any $m \geq 1$, whence u is a smooth solution of (5.3) away from Σ.

Next we want to extend u to be a weak solution in $\mathbb{R}^n \times [-1, 0]$. To achieve this, we first estimate n-dimensional parabolic Hausdorff measure of Σ.

For any compact set $K \subset \mathbb{R}^n \times (-1, 0)$ and $\rho_0 > 0$, since $K \cap \Sigma$ is compact, Vitali's covering lemma implies that there exist $1 \leq l < +\infty$, $\{z_i\} \subset K \cap \Sigma$ and $0 < r_i \leq r_0$ such that $\{Q_{r_i}(z_i)\}_{i=1}^l$ are mutally disjoint and $K \cap \Sigma \subset \bigcup_{1 \leq i \leq l} Q_{5r_i}(z_i)$. Denote $\bar{z}_i = z_i + (0, r_i^2)$. Then there exists $k \in \mathbb{N}$ such that for any $\delta \in (0, 1)$,

$$\begin{aligned}
\epsilon_0^2 &\leq \Phi_{z_i}^k(\delta r_i) \leq C \int_{t_i - 4\delta^2 r_i^2}^{t_i - \delta^2 r_i^2} \int_{\mathbb{R}^n} |\nabla u_k|^2 G_{z_i} \\
&\leq C(\delta) r_i^{-n} \int_{Q_{r_i}(z_i)} |\nabla u_k|^2 \, dxdt \\
&+ C\delta^{-n} \exp\left(-\frac{1}{16\delta^2}\right) \int_{t_i - 4\delta^2 r_i^2}^{t_i - \delta^2 r_i^2} \int_{\mathbb{R}^n} |\nabla u_k|^2 G_{\bar{z}_i} \\
&\leq C(\delta) r_i^{-n} \int_{Q_{r_i}(z_i)} |\nabla u_k|^2 + C\delta^{2-n} \exp\left(-\frac{1}{16\delta^2}\right) E_0
\end{aligned}$$

for $1 \leq i \leq l$, where we have used the fact

$$G_{z_i} \leq \delta^{-n} \exp\left(-\frac{1}{16\delta^2}\right) G_{\bar{z}_i} \ \text{ in } \ \mathbb{R}^n \times \left[t_i - 4\delta^2 r_i^2, t_i - \delta^2 r_i^2\right] \setminus Q_{r_i}(z_i),$$

and Theorem 7.1.3 in the last two steps. By choosing $\delta > 0$ sufficiently small, we have

$$C\delta^{2-n} \exp\left(-\frac{1}{16\delta^2}\right) E_0 \leq \frac{\epsilon_0^2}{2}$$

so that

$$r_i^n \leq C \int_{Q_{r_i}(z_i)} |\nabla u_k|^2 \, dxdt, \quad 1 \leq i \leq l.$$

Summing over $1 \leq i \leq l$, we obtain

$$\begin{aligned}
\mathcal{P}_{5\delta}^n(K \cap \Sigma) &\leq \sum_{1 \leq i \leq l} (5r_i)^n \leq C \sum_{1 \leq i \leq l} \int_{Q_{r_i}(z_i)} |\nabla u_k|^2 \, dxdt \\
&= C \int_{\bigcup_{1 \leq i \leq l} Q_{r_i}(z_i)} |\nabla u_k|^2 \, dxdt \leq CE_0.
\end{aligned}$$

Sending δ to zero, this implies that the n-dimensional parabolic Hausdorff measure of Σ is locally finite. This also implies that there is a finite cover $\{Q_{r_i}(z_i)\}_{1\le i\le J_\delta}$ of Σ, with $r_i < \delta$, such that

$$\left|\bigcup_{1\le i\le J_\delta} Q_{r_i}(z_i)\right| \to 0, \quad \text{as} \quad \delta \to 0,$$

where $|\cdot|$ is the Lebesgue measure on $\mathbb{R}^n \times \mathbb{R}$.
Let $\phi \in C_0^\infty(Q_2(0))$ be such that $0 \le \phi \le 1$ and $\phi \equiv 1$ on $Q_1(0)$. For $i \ge 1$, let $\phi_i(z) = \phi(\frac{x-x_i}{r_i}, \frac{t-t_i}{r_i^2}) \in C_0^\infty(Q_{2r_i}(z_i))$. Given $\psi \in C_0^\infty(\mathbb{R}^n \times (-1,0), \mathbb{R}^L)$, define $\eta = \psi \inf_i (1-\phi_i) \in C_0^\infty(\mathbb{R}^n \times (-1,0) \setminus \Sigma, \mathbb{R}^L)$. It is easy to see that $\eta \to \psi$ a.e. as $\delta \to 0$.

Since $\text{supp}(\eta) \subset \mathbb{R}^n \times (-1,0) \setminus \Sigma$, testing (5.3) with η gives

$$\int_{\mathbb{R}^n\times(-1,0)} (\partial_t u - \Delta u - A(u)(\nabla u, \nabla u))\,\eta = 0.$$

Hence

$$\begin{aligned}
&\int_{\mathbb{R}^n\times(-1,0)} (\partial_t u - \Delta u - A(u)(\nabla u, \nabla u))\,\psi \\
=\ &\int_{\mathbb{R}^n\times(-1,0)} (\partial_t u - \Delta u - A(u)(\nabla u, \nabla u))\,\psi \left(1 - \inf_i(1-\phi_i)\right) \\
\le\ & C\int_{\mathbb{R}^n\times(-1,0)} \left\{\left(|\partial_t u| + |\nabla u|^2\right)|\psi - \eta| + |\nabla u|\,|\nabla\psi| \left|1 - \inf_i(1-\phi_i)\right|\right\} \\
&+ \int_{\mathbb{R}^n\times(-1,0)} |\nabla u|\,|\psi| \left|\nabla(\inf_i(1-\phi_i))\right| \\
\le\ & C\int_{\mathbb{R}^n\times(-1,0)} |\nabla u||\psi| \left|\nabla\left(\inf_i(1-\phi_i)\right)\right| + o(1) \\
\le\ & C\,\|\nabla u\|_{L^2(\bigcup_i Q_{r_i}(z_i))} \left(\int_{\mathbb{R}^n\times(-1,0)} |\nabla(\inf_i(1-\phi_i))|^2\right)^{\frac12} + o(1) \\
\le\ & o(1)\left(\sum_i r_i^{-2}|Q_{r_i}(z_i)|\right)^{\frac12} + o(1) \to 0, \quad \text{as } \delta \to 0.
\end{aligned}$$

Hence u is a weak solution of (5.3). By the lower semicontinuity of $\Phi(r; u_k)$ under the weak convergence and (7.4), we have

$$\Phi(R; u) \le \lim_{k\to\infty} \Phi(R; u_k) \le \lim_{k\to\infty} \Phi(1; u_k) = \Phi(1; u), \quad \forall 0 < R \le 1.$$

It is worth to mention that the above size estimate of Σ implies that for a.e. $t \in (-1,0)$, $\Sigma_t = \Sigma \cap \{t\}$ has locally finite $(n-2)$-dimensional Hausdorff measure with respect to the Euclidean metric in $\mathbb{R}^n$. Cheng [26] has obtained a stronger estimate. More precisely,

Theorem 7.2.4 *For any $t_0 \in (-1, 0)$, it holds*

$$H^{n-2}(\Sigma_{t_0}) < +\infty. \tag{7.20}$$

Here we present his proof, which is based on the following lemma.

Lemma 7.2.5 *For any $\epsilon > 0$, there exists $K(\epsilon) > 0$ such that for any $t_0 \in (-1, 0)$ and $R \leq \sqrt{1 + t_0}$, the following estimate holds on $T_R(t_0)$:*

$$G_{z_0}(x,t) \leq \begin{cases} R^n, & \text{for all } x \in \mathbb{R}^n \\ \epsilon G_{z_0+(0,R^2)}(x,t), & \text{if } |x - x_0| \geq K(\epsilon). \end{cases}$$

Proof. The first inequality is trivial. To prove the second inequality. For $K = K(\epsilon) > 0$ to be chosen, if $|x - x_0| \geq K(\epsilon)R$, then we have

$$\begin{aligned} \frac{G_{z_0}}{G_{z_0+(0,R^2)}} &= \left(\frac{t_0 - t + R^2}{t_0 - t}\right)^{\frac{n}{2}} \exp\left(\frac{|x-x_0|^2}{4(t_0 - t + R^2)} - \frac{|x - x_0|^2}{4(t_0 - t)}\right) \\ &\leq 5^{\frac{n}{2}} \exp\left(-\frac{R^2|x - x_0|^2}{4(t_0 - t + R^2)(t_0 - t)}\right) \\ &\leq 5^{\frac{n}{2}} \exp\left(-\frac{K^2}{80}\right) \leq \epsilon \end{aligned}$$

provided that $K(\epsilon)$ is chosen to be sufficiently large. □

Proof of Theorem 7.2.4:
By Lemma 7.2.5, we have

$$\begin{aligned} \int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0} &\leq R^{-n} \int_{t_0-4R^2}^{t_0-R^2} \int_{B_{KR}(x_0)} |\nabla u_k|^2 \\ &+ \epsilon \int_{t_0-4R^2}^{t_0-R^2} \int_{\mathbb{R}^n} |\nabla u_k|^2 G_{z_0+(0,R^2)}. \end{aligned}$$

On the other hand,

$$\begin{aligned} &\epsilon \int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0+(0,R^2)}\, dxdt \\ = &\ 2\epsilon \int_{t_0-4R^2}^{t_0-R^2} (R^2 + t_0 - t)^{-1} \Phi^k_{z_0+(0,R^2)}(\sqrt{R^2 + t_0 - t})\, dt \\ \leq &\ 2\epsilon \Phi^k_{z_0+(0,R^2)}(\sqrt{t_0 + R^2}) \int_{t_0-4R^2}^{t_0-R^2} (R^2 + t_0 - t)^{-1}\, dt \\ \leq &\ C\epsilon(t_0 + R^2)^{1-\frac{n}{2}} \left(\int_{\mathbb{R}^n} |\nabla u_k|^2\, dx\right) \\ \leq &\ C\epsilon t_0^{1-\frac{n}{2}} E_0 \leq \frac{1}{2}\epsilon_0 \end{aligned}$$

provided that $\epsilon = \epsilon(\epsilon_0, E_0, t_0) > 0$ is chosen to be sufficiently small.

For $0 < R < \sqrt{t_0}$, denote

$$\Sigma_{t_0} = \bigcap_{R>0} \Sigma_{t_0}^R, \text{ where } \Sigma_{t_0}^R = \left\{ x \in \mathbb{R}^n \mid \lim_{k\to\infty} \int_{T_R(t_0)} |\nabla u_k|^2 G_{t_0}\, dxdt \right\}.$$

For any compact $K \subset \mathbb{R}^n$, if $x_0 \in \Sigma_{t_0}^R \cap K$, then we have

$$\begin{aligned} \epsilon_0 &\le \lim_{k\to\infty} \int_{T_R(t_0)} |\nabla u_k|^2 G_{z_0} \\ &\le \frac{1}{2}\epsilon_0 + \lim_{k\to\infty} R^{-n} \int_{B_{KR}(x_0)\times[t_0-4R^2,t_0-R^2]} |\nabla u_k|^2 . \end{aligned}$$

Hence

$$R^n \le \frac{2}{\epsilon_0} \lim_{k\to\infty} \int_{B_{KR}(x_0)\times[t_0-4R^2,t_0-R^2]} |\nabla u_k|^2 . \tag{7.21}$$

Note that, since $\Sigma_{t_0}^R \cap K$ is compact, there is a finite subfamily $\mathcal{J}_1 = \{B_{KR}(x_j)\}$ of the family $\mathcal{J} = \{B_{KR}(x_0) \mid x_0 \in \Sigma_{t_0}^R \cap K\}$ such that any two balls in $\mathcal{J}_1$ are disjoint and $\{B_{5KR}(x_j)\}$ covers $\Sigma_{t_0}^R \cap K$. Thus

$$\begin{aligned} \sum_j (5KR)^n &= = (5K)^n \sum_j R^n \\ &\le \frac{2(5K)^n}{\epsilon_0} \sum_j \lim_{k\to\infty} \int_{B_{KR}(x_j)\times[t_0-4R^2,t_0-R^2]} |\nabla u_k|^2 \\ &\le \frac{2(5K)^n}{\epsilon_0} \lim_{k\to\infty} \int_{\mathbb{R}^n\times[t_0-4R^2,t_0-R^2]} |\nabla u_k|^2 \\ &\le CK^n E_0 R^2. \end{aligned}$$

Therefore

$$H^{n-2}(\Sigma_{t_0} \cap K) = \lim_{R\to 0} H_R^{n-2}(\Sigma_{t_0}^R \cap K) \le C(K, \epsilon_0) < +\infty.$$

Since K is arbitrary, this implies (7.20). □

We now discuss an asymptotic behavior of the short time smooth solution of (5.3) near a type I singularity.

Definition 7.2.6 Let $0 < T < +\infty$ be the first singular time of a smooth solution $u \in C^\infty(\mathbb{R}^n \times (0, T), N)$ of (5.3), with an isolated singularity at $(0, T)$. If

$$|\nabla u|(x, t) \le \frac{C}{\sqrt{T-t}}, \quad \forall x \in \mathbb{R}^n, 0 < t < T, \tag{7.22}$$

then $(0, T)$ is called to be a type I singularity.

Definition 7.2.7 A nontrivial, smooth solution $u \in C^\infty(\mathbb{R}^n \times (-\infty, 0), N)$ of (5.3) is called to be self-similar, if

$$u(x,t) = u(\frac{x}{\sqrt{-t}}), \;\; x \in \mathbb{R}^n \text{ and } \; t \in (-\infty, 0).$$

Proposition 7.2.8 *For a smooth solution $u \in C^\infty(\mathbb{R}^n \times (0,T), N)$ of (5.3) with $(0,T)$ being a type I singularity. For $R > 0$, define*

$$u_R(x,t) = u(Rx, T + R^2 t) : \mathbb{R}^n \times (-\frac{T}{R^2}, 0) \to N.$$

Then there exists a self-similar solution $\phi \in C^\infty(\mathbb{R}^n \times (-\infty,0), N)$ of (5.3) such that after taking possible subsequences,

$$u_R \to \phi \text{ in } C^k_{\text{loc}}(\mathbb{R}^n \times (-\infty, 0), N) \text{ for any } k \geq 1 \text{ as } R \to 0.$$

Proof. First observe that for any $R > 0$, we have

$$|\nabla u_R|(x,t) \leq \frac{C}{\sqrt{|t|}}, \;\; \forall x \in \mathbb{R}^n \text{ and } t \in \left(-\frac{T}{R^2}, 0\right).$$

Therefore, for any compact subset $K \subset \mathbb{R}^n \times (-\infty, 0)$, there exists a sufficiently small $R_0 = R_0(K) > 0$ such that

$$\max_K |\nabla u_R| \leq C(K) < +\infty, \quad \forall R \geq R_0.$$

Higher order derivatives of u_R are also bounded on K. Therefore we may assume that there exists a smooth solution $\phi \in C^\infty(\mathbb{R}^n \times (-\infty, 0), N)$ of (5.3) such that after passing to subsequences, $u_R \to \phi$ in $C^k_{\text{loc}}(\mathbb{R}^n \times (-\infty, 0), N)$ for any $k \geq 1$, as $R \to 0$. Since $(0,T)$ is a singularity of u, $\phi \not\equiv$ constant. In fact, if ϕ is a constant, then fix some small $R > 0$, we would have $\Phi_{(0,T)}(R; u) = \Phi(1; u_R) \leq \epsilon_0$ and hence $(0,T)$ is a regular point of u. To see that ϕ is a self-similar. By (7.4), we have

$$\begin{aligned}
&\int_{-T_2}^{-T_1} \int_{\mathbb{R}^n} \frac{|2t\partial_t \phi + x \cdot \nabla \phi|^2}{2|t|} G \, dxdt \\
&\leq \lim_{R \to 0} \int_{-T_2}^{-T_1} \int_{\mathbb{R}^n} \frac{|2t\partial_t u_R + x \cdot \nabla u_R|^2}{2|t|} G \, dxdt \\
&= \lim_{R \to 0} \int_{T - T_2 R^2}^{T - T_1 R^2} \int_{\mathbb{R}^n} \frac{|2t\partial_t u + x \cdot \nabla u|^2}{2|t|} G \, dxdt = 0
\end{aligned}$$

for any $0 < T_1 < T_2 < +\infty$. It follows that

$$2t\partial_t \phi + x \cdot \nabla \phi = 0, \;\; \forall x \in \mathbb{R}^n \text{ and } t \in (-\infty, 0).$$

That is $\phi(x,t) = \phi(\frac{x}{\sqrt{-t}})$ and ϕ is self-similar. □

7.3 Finite time singularity in dimensions at least three

In this section, we will present the examples by Coron-Ghidaglia [39] and Chen-Ding [28] on the existence of finite time singularity for the heat flow of harmonic maps in higher dimensions.

The ideas of [39] and [28] are based on both the ϵ_0-apriori estimate and the existence of smooth maps ϕ from M to N which has trivial energy $E(\phi)$ with nontrivial topology. This is related to the following important theorem by White [210, 211]. Namely,

Theorem 7.3.1 *For compact Riemannian manifolds M, N and $u_0 \in C^\infty(M,N)$. Then*

$$\inf\left\{E(u) \mid u \in C^\infty(M,N), u \text{ is homotopic to } u_0\right\} > 0$$

iff the restriction of u_0 to a 2-skeleton of M is not homotopic to constant.

Remark 7.3.2 If one of the following three properties holds
(a) $\pi_1(M) = \pi_2(M) = \{0\}$;
(b) $\pi_1(N) = \pi_2(N) = \{0\}$;
(c) $\pi_1(M) = \pi_2(N) = \{0\}$,
then for any $u_0 \in C^\infty(M,N)$,

$$\inf\left\{E(u) \mid u \in C^\infty(M,N), u \sim u_0\right\} = 0.$$

A typical example of such a u_0 is the Hopf map from S^3 to S^2. Recall that the Hopf map $H : S^3 \subset \mathbb{C}\times\mathbb{C} \to S^2 \subset \mathbb{C}\times\mathbb{R}$ is defined by

$$H(z,w) = \left(2z\bar{w}, |z|^2 - |w|^2\right), \quad (z,w) \in \mathbb{C}\times\mathbb{C}.$$

Then we have

$$\inf\left\{E(u) \mid u \in C^\infty\left(S^3,S^2\right), u \sim H\right\} = 0.$$

The following theorem is essentially due to [28].

Theorem 7.3.3 *For any $T > 0$ there exists a constant $\epsilon = \epsilon(M,N,T) > 0$ such that for any map $u_0 \in C^\infty(M,N)$ which is not homotopic to constant but satisfies $E(u_0) < \epsilon$, then any smooth solution u of (5.3) with initial data u_0 must blow up before $2T$.*

Proof. The argument here is a simplied version of [28] given by [196]. Suppose that $u \in C^\infty(M \times [0,2T],N)$ solves (5.3) with $u(x,0) = u_0(x)$. For $\bar{z} = (\bar{x},\bar{t})$, $T \le \bar{t} \le 2T$, $R^2 = T$, we have

$$\Phi_{\bar{z}}(R) \le CR^{2-n}E\left(u\left(\bar{t}-R^2\right)\right) \le CR^{2-n}E_0 < \epsilon_0^2$$

if $\epsilon < \frac{\epsilon_0 T^{\frac{n-2}{2}}}{C}$, where $C = C(M,N) > 0$ and $\epsilon_0 = \epsilon_0(M,N) > 0$ are given by Proposition 7.1.4. Hence we have

$$|\nabla u|(z) \le \frac{C}{R} = C(M,N,T), \ \forall z \in M \times [T,2T].$$

By (5.3.3), we have

$$(\partial_t - \Delta_g)\, e(u) \leq Ce(u) \text{ on } M \times [T, 2T]$$

so that

$$\max_x e(u)\,(x, 2T) \leq CE(u_0) \leq C\epsilon$$

for some $C = C(M, N, T) > 0$. Thus if $\epsilon = \epsilon(M, N, T)$ is chosen to be sufficiently small, then the image of $u(2T)$ is contained in a contractible coordinate neighborhood of a point $p \in N$. Hence $u(2T)$ is homotopic to constant. This contradicts the choice of u_0. Therefore u must blow up before $2T$. □

7.4 Nonuniqueness of heat flow of harmonic maps

In this section, we will present the example by Coron [38] in which the heat flow of harmonic maps has infinitely many weak solutions. Interested readers can also see Bethuel-Coron-Ghidaglia-Soyeur [12] for related results.

The crucial observation in [38] is that there exists a weakly harmonic map $u_0 : B^3 \to S^2$ such that $u(x,t) = u_0(x), (x,t) \in B^3 \times (0, +\infty)$, viewed as a static solution of (5.3), doesn't satisfy (7.4) and hence is different from the solution constructed in [33] (see §7.5 below).

Suppose that $u_0 \in W^{1,2}_{\text{loc}}(\mathbb{R}^3, S^2)$ is a weakly harmonic map that is homogeneous of degree zero, i.e., $u_0(x) = u_0(\frac{x}{|x|})$. Consider $u(x,t) = u_0(x)$. Then u weakly solves (5.3) and

$$\Phi_{\bar{z}}(\rho) = \frac{1}{(4\pi)^{\frac{3}{2}}\rho} \int_{\mathbb{R}^3} |\nabla u_0|^2 \exp\left(-\frac{|x-\bar{x}|^2}{4\rho^2}\right) dx < \infty$$

for any $\bar{z} \in \mathbb{R}^3 \times (0, +\infty)$ and $\rho > 0$.

Suppose that u satisfies (7.4). Then we have

$$\frac{1}{\rho}\int_{\mathbb{R}^3} |\nabla u_0|^2 \exp\left(-\frac{|x-\bar{x}|^2}{4\rho^2}\right) \leq \frac{1}{r}\int_{\mathbb{R}^3} |\nabla u_0|^2 \exp\left(-\frac{|x-\bar{x}|^2}{4r^2}\right) \tag{7.23}$$

for any $0 < \rho \leq r < +\infty$.

We now recall a simple lemma that characterizes the stationarity of weakly harmonic maps of homogeneous degree zero, due to Hardt [81].

Lemma 7.4.1 *Suppose that $\phi \in C^\infty(S^2, S^2)$ is a harmonic map. Then $\phi(\frac{x}{|x|}) : \mathbb{R}^3 \to S^2$ is stationary iff*

$$q := \int_{S^2} |\nabla\phi|^2(x) x \, dH^2(x) = 0. \tag{7.24}$$

Let $\pi : S^2 \setminus \{(0,0,1)\} \to \mathbb{R}^2 \cong \mathbb{C}$ be the stereographic projection from the north pole of S^2. Then the following fact is well-known (see for example [19]).

Proposition 7.4.2 *$\phi : S^2 \to S^2$ is a harmonic map iff $\psi(z) := \pi\left(\phi(\pi^{-1}(z))\right) : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ is a rational function.*

Remark 7.4.3 For $\lambda \in \mathbb{R}$, define $g_\lambda(z) = \lambda z : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$. Then

$$u_\lambda(x) = \pi^{-1}\left(g_\lambda\left(\pi\left(\frac{x}{|x|}\right)\right)\right) : \mathbb{R}^3 \to S^2 \tag{7.25}$$

is a stationary harmonic map iff $\lambda = 1$.

Now we have

Theorem 7.4.4 *If $u_0(x) = u_\lambda(x) : \mathbb{R}^3 \to S^2$ for some $\lambda \neq 1$, then there exist infinitely many weak solutions $u : \mathbb{R}^3 \times (0, +\infty) \to S^2$ of (5.3) with the initial data u_0.*

Proof. Denote

$$\phi(\rho, \bar{x}) = \frac{1}{\rho}\int_{\mathbb{R}^3} |\nabla u_0|^2 \exp\left(-\frac{|x-\bar{x}|^2}{4\rho^2}\right), \ \bar{x} \in \mathbb{R}^3.$$

Note that

$$\phi(\rho, 0) = \int_0^\infty \left(\int_{S^2} |\nabla u_0|^2 \, dH^2\right) \exp\left(-\frac{r^2}{4\rho^2}\right) \frac{dr}{\rho} = a_0$$

is independent of $\rho > 0$. Moreover, direct calculations imply

$$\begin{aligned}
\nabla_{\bar{x}}\phi(\rho, 0) &= \int_{\mathbb{R}^3} |\nabla u_0|^2(x) \frac{x}{2\rho^3} \exp\left(-\frac{|x|^2}{4\rho^2}\right) \\
&= \int_0^\infty \left(\int_{S^2} |\nabla u_0|^2(\sigma)\sigma \, dH^2(\sigma)\right) \cdot \frac{\exp\left(-\frac{r^2}{4\rho^2}\right)}{2\rho^3} r \, dr \\
&= \frac{1}{\rho}\left(\int_0^\infty \exp(-r) \, dr\right) \cdot \left(\int_{S^2} |\nabla u_0|^2(\sigma)\sigma \, dH^2(\sigma)\right) \\
&= \frac{1}{\rho}\int_{S^2} |\nabla u_0|^2(\sigma)\sigma \, dH^2(\sigma) := \frac{q}{\rho}.
\end{aligned}$$

Hence for $0 < \rho_1 < \rho_2$, if $t > 0$ is sufficiently small, then we have

$$\phi(\rho_1, tq) = a_0 + t\frac{|q|^2}{\rho_1} + O(t^2) > \phi(\rho_2, tq) = a_0 + t\frac{|q|^2}{\rho_2} + O(t^2),$$

this contradicts (7.23). On the other hand, the solution $\tilde{u}$ of (5.3) with the initial data u_0 constructed by [33] satisfies (7.23). Hence $\tilde{u} \neq u_0$. This yields non uniqueness in the energy class of weak solutions of (5.3) with initial data u_0.

Using $\tilde{u}$ and u_0, one can construct infinitely many weak solutions to (5.3) as follows. For any $T > 0$, define

$$u_T(x,t) = \begin{cases} u_0(x) & \text{if } \ 0 \leq t \leq T \\ \tilde{u}(x,t) & \text{if } \ t > T. \end{cases}$$

It is easy to verify that $u_T, T > 0$ is a family of infinitely many weak solutions of (5.3). □

Remark 7.4.5 (a) The solution $\tilde{u}$ with initial data u_0 can't be of the form $\tilde{u}(x,t) = v(\frac{x}{|x|}, t)$. For, otherwise, v would be a weak solution of (5.3) on $S^2 \times (0, +\infty)$. Note that $u_0 \in C^\infty(S^2, S^2)$ is harmonic. Hence, by the local unique solvability theorem of (5.3) on $S^2 \times (0, +\infty)$ for smooth initial data, we would have $\tilde{u} \equiv u_0$. This is proved to be impossible.
(b) It is an open problem *to find the class of weak solutions of* (5.3) in which the uniqueness does hold. It is unknown whether the uniqueness holds for the class of weak solutions satisfying the monotonicity inequality.

$$\Phi_{\bar{z}}(\rho) \le \Phi_{\bar{z}}(r)$$

for all $\bar{z} \in \mathbb{R}^n \times (0, +\infty)$ and $0 < \rho \le r \le \sqrt{\bar{t}}$. See Hong [96] for some related works.

7.5 Global weak heat flows into spheres

In this section, we will present the existence of global weak solutions to (5.3) for $N = S^{L-1} \subset \mathbb{R}^L$, due to Chen [27], in which the Ginzburg-Landau approximation scheme was introduced to establish the existence of approximate weak solutions of (5.3). The symmetry of S^{L-1} plays a critical role in the convergence. Similar observations have been made by Shatah [180] for wave maps and Keller-Rubinstein-Sternberg [110] for singular perturbation problems.

The main theorem of this section is

Theorem 7.5.1 *For any $u_0 \in W^{1,2}(M, S^{L-1})$, there is a weak solution $u : M \times \mathbb{R}_+ \to S^{L-1}$, with $\nabla u \in L^\infty(\mathbb{R}_+, L^2(M))$ and $\partial_t u \in L^2(\mathbb{R}_+, L^2(M))$, to*

$$\partial_t u - \Delta u = |\nabla u|^2 u, \quad \text{in } M \times \mathbb{R}_+ \tag{7.26}$$

$$u|_{t=0} = u_0, \quad \text{on } M. \tag{7.27}$$

For simplicity, we consider the case $M = \mathbb{R}^n$ only. The Ginzburg-Landau approximation scheme is as follows. For $\epsilon > 0$, consider the Cauchy problem for maps $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^L$:

$$\partial_t u - \Delta u = \frac{1}{\epsilon^2}\left(1 - |u|^2\right) u, \quad \text{in } \mathbb{R}^n \times \mathbb{R}_+ \tag{7.28}$$

$$u|_{t=0} = u_0, \quad \text{on } \mathbb{R}^n. \tag{7.29}$$

Note that (7.28) is the negative L^2-gradient flow of the Ginzburg-Landau energy functional

$$E_\epsilon(u) = \int_{\mathbb{R}^n} \left(\frac{1}{2}|\nabla u|^2 + \frac{1}{4\epsilon^2}(1 - |u|^2)^2\right).$$

Write $u(t)(x) = u(x,t) : \mathbb{R}^n \to \mathbb{R}^L$ for $t \ge 0$. Then we have

Lemma 7.5.2 *For $0 < T \le \infty$, if $u \in C^\infty(\mathbb{R}^n \times (0,T], \mathbb{R}^L)$ solves (7.28) and (7.29), then*

$$E_\epsilon(u(t)) + \int_0^t \int_{\mathbb{R}^n} |\partial_t u|^2 \, dxdt = E_\epsilon(u_0) = E(u_0), \; 0 < t \le T. \tag{7.30}$$

In particular, u attains its initial data continuously in $W^{1,2}(\mathbb{R}^n, S^{L-1})$.

Proof. Multiplying (7.28) by $\partial_t u$ and integrating over $\mathbb{R}^n \times (0,t)$ gives (7.5.2). Since $\partial_t u \in L^2(\mathbb{R}^n \times [0,t])$, it is clear that $u(t) \to u_0$ weakly in $W^{1,2}(\mathbb{R}^n, S^{L-1})$ as $t \downarrow 0$. By the lower semicontinuity, we have

$$E(u_0) \leq \liminf_{t \downarrow 0} E(u(t)).$$

On the other hand, by (7.5.2) we have

$$\limsup_{t \downarrow 0} E(u(t)) \leq \lim_{t \downarrow 0} E_\epsilon(u(t)) \leq E(u_0).$$

Thus $u(t)$ converges to u_0 strongly in $W^{1,2}(\mathbb{R}^n, S^{L-1})$ as $t \downarrow 0$. □

Next we have the L^∞-estimate.

Lemma 7.5.3 *For $0 < T \leq \infty$, if $u \in C^\infty(\mathbb{R}^n \times (0,T], \mathbb{R}^L)$ solves (7.28) and (7.29), then*

$$\|u\|_{L^\infty(\mathbb{R}^n \times (0,T])} \leq 1.$$

Proof. Multiplying (7.28) by u, we obtain

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left(\frac{|u|^2}{2}\right) + |\nabla u|^2 + \frac{1}{\epsilon^2}\left(|u|^2 - 1\right)|u|^2 = 0.$$

In particular,

$$\left(\frac{\partial}{\partial t} - \Delta + \frac{2}{\epsilon^2}\right)\left(|u|^2 - 1\right) \leq 0.$$

The conclusion follows from the maximum principle, since $|u(x,0)|^2 - 1 = |u_0|^2 - 1 = 0$. □

By using Galerkin's method, one can show that for any $\epsilon > 0$, there is a unique smooth solution $u_\epsilon \in C^\infty((0,+\infty), \mathbb{R}^L)$ of (7.28) and (7.29). Moreover, by Lemma 7.5.2 and 7.5.3 we have

$$\int_0^{+\infty} \int_{\mathbb{R}^n} |\partial_t u_\epsilon|^2 + \sup_{0<t<\infty} E\left(u_\epsilon(t)\right) \leq E(u_0),$$

$$|u_\epsilon| \leq 1, \quad \sup_{0<t<\infty} \left\|1 - |u_\epsilon(t)|^2\right\|_{L^2(\mathbb{R}^n)} \leq 4\epsilon^2 E(u_0).$$

Thus, up to a subsequence, we may assume that there exists $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^L$, with $\partial_t u \in L^2(\mathbb{R}^n \times \mathbb{R}_+)$ and $\nabla u \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^n))$, such that

$$u_\epsilon \to u \text{ in } L^2_{\text{loc}}\left(\mathbb{R}^n \times \mathbb{R}_+, \mathbb{R}^L\right),$$

$$\nabla u_\epsilon \to \nabla u \text{ weak}^* \text{ in } L^\infty\left(\mathbb{R}_+, L^2(\mathbb{R}^n)\right),$$

$$\partial_t u_\epsilon \to \partial_t u \text{ weakly in } L^2\left(\mathbb{R}^n \times \mathbb{R}_+\right).$$

In particular, $|u| = 1$. To show that u is a weak solution of (5.3), we need to explore the symmetry of S^{L-1}.

Lemma 7.5.4 *A map* $u : \mathbb{R}^n \times \mathbb{R}_+ \to S^{L-1}$*, with* $\nabla u \in L^\infty(\mathbb{R}_+, L^2(\mathbb{R}^n))$ *and* $\partial_t u \in L^2(\mathbb{R}^n \times \mathbb{R}_+)$*, is a weak solution of (7.26) iff*

$$u_t \wedge u - div(\nabla u \wedge u) = 0 \quad in \quad \mathbb{R}^n \times \mathbb{R}_+, \tag{7.31}$$

where $\wedge$ *denotes the wedge product in* $\mathbb{R}^L$*.*

Proof. If u weakly solves (7.26), then for any $\phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R})$, multiplying (7.26) by $\wedge(u\phi)$ and integration gives

$$\int_{\mathbb{R}^n \times \mathbb{R}_+} (u_t \wedge (u\phi) + \nabla u \wedge \nabla(u\phi)) = 0,$$

since $|\nabla u|^2 u \wedge (u\phi) = 0$. Note also that

$$\int_{\mathbb{R}^n \times \mathbb{R}_+} \nabla u \wedge \nabla(u\phi) = \int_{\mathbb{R}^n \times \mathbb{R}_+} (\nabla u \wedge u) \cdot \nabla\phi.$$

Hence u weakly solves (7.31).

If u weakly solves (7.31). Then we have

$$(\partial_t u - \Delta u) \wedge u = 0.$$

Hence there is a multiplier $\lambda : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ such that

$$\partial_t u - \Delta u = \lambda u \tag{7.32}$$

Multiplying it by $u\phi$ and using $\partial_t u \cdot u = 0$ and $\Delta u \cdot u = -|\nabla u|^2$, we obtain that for any $\phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$,

$$\int_{\mathbb{R}^n \times \mathbb{R}_+} \lambda\phi = \int_{\mathbb{R}^n \times \mathbb{R}_+} |\nabla u|^2 \phi.$$

Therefore u weakly solves (7.31). □

For $\phi \in C_0^\infty(\mathbb{R}^n \times \mathbb{R}_+)$, taking wedge product of (7.28) with $u_\epsilon \phi$ and integrating over $\mathbb{R}^n \times \mathbb{R}_+$, we obtain

$$\int_{\mathbb{R}^n \times \mathbb{R}_+} (\partial_t u_\epsilon \wedge u_\epsilon \phi + \nabla u_\epsilon \wedge u_\epsilon \cdot \nabla\phi) = 0.$$

Since

$$\partial_t u_\epsilon \wedge u_\epsilon \to \partial_t u \wedge u \text{ and } \nabla u_\epsilon \wedge u_\epsilon \to \nabla u \wedge u \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}_+)$$

as $\epsilon \to 0$, u weakly solves (7.31) and hence (7.26). □

7.6 Global weak heat flows into general manifolds

In this section, we will present the work by [33] on the existence of global, weak solutions of (5.3) for any compact Riemannian manifold $N \subset \mathbb{R}^L$. The main difficulty comes from the fact that N doesn't have symmetry property. A crucial observation by [33] is that the Ginzburg-Landau approximate solutions enjoy uniform estimates under the small condition on the normalized energy. First we need to modify the Ginzburg-Landau energy functional E_ϵ. Let $\delta_N > 0$ be such that the nearest point projection $\Pi_N : N_{2\delta_N} \to N$ is smooth and $\text{dist}^2(p, N)$ is smooth for $p \in N_{2\delta_N}$.

Throughout the discussion henceforth, let χ be a smooth, monotonically increasing function such that

$$\chi(s) = \begin{cases} s & \text{for } s \le \delta_N^2 \\ 4\delta_N^2 & \text{for } s \ge 4\delta_N^2. \end{cases}$$

Define $F \in C^\infty(\mathbb{R}^L, \mathbb{R}_+)$ by

$$F(p) = \chi(\text{dist}^2(p, N)), \quad \text{and } f(p) = -\nabla_p F(p).$$

It is clear that for $p \in N_{\delta_N}$, $F(p) = \text{dist}^2(p, N)$.

For $\epsilon > 0$, define

$$E_\epsilon(u) \equiv \int_M e_\epsilon(u) = \int_M \left(\frac{1}{2} |\nabla u|^2 + \frac{1}{\epsilon^2} F(u) \right). \tag{7.33}$$

For $\phi \in C^\infty(M, N)$, the Ginzburg-Landau heat flow becomes

$$\partial_t u - \Delta u = \frac{1}{\epsilon^2} f(u) \ \text{ in } \ M \times \mathbb{R}_+ \tag{7.34}$$

$$u|_{t=0} = \phi \text{ on } \ M. \tag{7.35}$$

By Galerkin's method, for any $\epsilon > 0$ there exists a smooth solution $u_\epsilon \in C^\infty(M \times \mathbb{R}_+, \mathbb{R}^L)$ of (7.34). As in Lemma 7.5.2, we have the following energy inequality.

Lemma 7.6.1 *For $\epsilon > 0$, u_ϵ satisfies*

$$\int_0^\infty \int_M |\partial_t u|^2 + \sup_{0<t<+\infty} E_\epsilon(u(t)) \le \frac{1}{2} \int_M |\nabla \phi|^2 . \tag{7.36}$$

We now want to show that u_ϵ satisfies an energy monotonicity inequality similar to (7.4) and (7.7).

Let $i_M > 0$ be the injectivity radius of M. For $x_0 \in M$, let $\phi \in C_0^\infty(B_{i_M}(x_0))$ be such that $0 \le \phi \le 1$, $\phi = 1$ in $B_{\frac{i_M}{2}}(x_0)$. For $z_0 = (x_0, t_0) \in M \times (0, +\infty)$ and $0 < R < \min\{i_M, \sqrt{t_0}\}$, introduce

$$\Phi(u, z_0, R) = R^2 \int_{S_R(t_0)} e_\epsilon(u) G_{z_0} \phi^2 \, dv_g,$$

$$\Psi(u, z_0, R) = \int_{T_R(t_0)} e_\epsilon(u) G_{z_0} \phi^2 \, dv_g dt.$$

Lemma 7.6.2 *Suppose that $u = u_\epsilon : M \times \mathbb{R}_+ \to \mathbb{R}^L$ is a smooth solution to (7.34), with $E(u(t)) \le E_0$. Then there exists a constant $c > 0$ depending only on M and N such that for any $z_0 = (x_0, t_0) \in M \times (0, +\infty)$ and $0 < R \le R_0 \le \min\{i_M, \sqrt{t_0}\}$, there holds*

$$\Phi(u, z_0, R) \le e^{c(R_0 - R)}\Phi(u, z_0, R_0) + cE_0 (R_0 - R), \tag{7.37}$$

$$\Psi(u, z_0, R) \le e^{c(R_0 - R)}\Psi(u, z_0, R_0) + cE_0 (R_0 - R). \tag{7.38}$$

Proof. For simplicity, assume $z_0 = (0, 0)$ and $u : M \times \mathbb{R}_- \to \mathbb{R}^L$ solves (7.34). For $R > 0$, define $u_R(x, t) = u(Rx, R^2 t)$. Then after scalings, we have

$$\Psi(u, (0,0), R) = \int_{T_1} \left(\frac{1}{2} g^{\alpha\beta}(R\cdot) \frac{\partial u_R}{\partial x_\alpha}\frac{\partial u_R}{\partial x_\beta} + \frac{R^2}{\epsilon^2} F(u_R)\right) \cdot G\phi^2(R\cdot)\sqrt{|g|}(R\cdot)\, dxdt.$$

Using $\nabla G = \frac{x}{2t} G$ and $\frac{d}{dR}|_{R=1} u_R(x, t) = x \cdot \nabla u + 2t\partial_t u$, we compute

$$\begin{aligned}
\frac{d}{dR}|_{R=1}\Psi (u, (0,0), R) &= \int_{T_1} [\left(-\frac{x_\beta}{2t} g^{\alpha\beta}\frac{\partial u}{\partial x_\alpha} - \Delta u + \frac{1}{\epsilon^2} f(u)\right) \cdot (x \cdot \nabla u + 2t\partial_t u) \\
&+ \frac{2}{\epsilon^2} F(u)] G\phi^2 \sqrt{g}\, dxdt \\
&- 2\int_{T_1} g^{\alpha\beta} \frac{\partial u}{\partial x_\alpha}\frac{\partial \phi}{\partial x_\beta} (x \cdot \nabla u + 2t\partial_t u)\, G\phi\sqrt{g}\, dxdt \\
&+ \frac{1}{2}\int_{T_1} x \cdot \nabla g^{\alpha\beta} \frac{\partial u}{\partial x_\alpha}\frac{\partial u}{\partial x_\beta} G\phi^2\sqrt{g}\, dxdt \\
&+ \frac{1}{2}\int_{T_1} e_\epsilon(u) G\phi^2 \frac{x \cdot \nabla g}{g}\sqrt{g}\, dxdt \\
&+ \int_{T_1} e_\epsilon(u) G\phi x \cdot \nabla\phi\sqrt{g}\, dxdt \\
&= I_1 + I_2 + I_3 + I_4 + I_5.
\end{aligned}$$

I_1 can be estimated by

$$\begin{aligned}
I_1 &\ge \int_{T_1} \left(\frac{1}{2|t|} |x \cdot \nabla u + 2t\partial_t u|^2 + \frac{2}{\epsilon^2} F(u)\right) G\phi^2\sqrt{g}\, dxdt \\
&- \int_{T_1} \frac{1}{2|t|} |x|\, |g - \mathrm{id}|\, |\nabla u|\, |x \cdot \nabla u + 2t\partial_t u|\, G\phi^2\sqrt{g}\, dxdt \\
&\ge \int_{T_1} \left(\frac{1}{4|t|} |x \cdot \nabla u + 2t\partial_t u|^2 + \frac{2}{\epsilon^2} F(u)\right) G\phi^2\sqrt{g}\, dxdt \\
&- c\int_{T_1} |x|^2\, |g - \mathrm{id}|^2\, |\nabla u|^2\, G\phi^2\sqrt{g}\, dxdt \\
&\ge \frac{1}{4}\int_{T_1} \frac{|x \cdot \nabla u + 2t\partial_t u|^2}{|t|} G\phi^2\sqrt{g}\, dxdt - c\Psi(u, (0,0), 1),
\end{aligned}$$

and the other terms can be estimated by

$$\begin{aligned}
|I_2| &\le \frac{1}{8}\int_{T_1} \frac{|x \cdot \nabla u + 2t\partial_t u|^2}{|t|} G\phi^2\sqrt{g}\, dxdt + c\int_{T_1} |\nabla u|^2 G\phi^2\sqrt{g}\, dxdt \\
&\le \frac{1}{2} I_1 + c\Psi(u, (0,0), 1) + cE_0.
\end{aligned}$$

$$|I_3| + |I_4| \le c\Psi\left(u, (0,0), 1\right),$$

$$|I_5| \le \frac{1}{2}\Psi\left(u, (0,0), 1\right) + cE_0.$$

For $0 < R_1 < 1$, since

$$\frac{d}{dR_1}\Psi\left(u, (0,0), R_1\right) = R_1^{-1}\frac{d}{dR}|_{R=1}\Psi\left(u, (0,0), RR_1\right),$$

$$|g(x) - \mathrm{id}| \le c|x|,$$

and

$$R_1^{-1}|t|^{-1}|x|^4 G \le G + c \text{ on } T_{R_1},$$

we can control the error term in I_1:

$$\int_{T_{R_1}} \frac{1}{2|t|}|x|^2\left|g - \mathrm{id}\right|^2\left|\nabla u\right|^2 G\phi^2\sqrt{g}\, dxdt \le c\Psi\left(u, (0,0), R_1\right) + cE_0.$$

The estimates for $I_2, \cdots, I_5$ can be done in similar ways. Hence we obtain

$$\begin{aligned}\frac{d}{dR}\Psi\left(u, (0,0), R\right) \;\ge\; & \frac{1}{8R}\int_{T_R}\frac{|x\cdot\nabla u + 2t\partial_t u|^2}{|t|}G\phi^2\sqrt{g}\, dxdt \\ & -c\Psi\left(u, (0,0), R\right) - cE_0\end{aligned}$$

This implies (7.38). □

For (7.34), we have a Bochner type inequality similar to (5.3.3).

Lemma 7.6.3 *Suppose that $u = u_\epsilon \in C^\infty(M \times \mathbb{R}_+, \mathbb{R}^L)$ solves (7.34). Then*

$$\left(\partial_t - \Delta\right) e_\epsilon(u) \le c\left(1 + e_\epsilon(u)\right) e_\epsilon(u) \tag{7.39}$$

where $c > 0$ is a constant depending only on M and N.

Proof. For simplicity, assume $M = \mathbb{R}^n$. Since for $\epsilon > 0$, $v(x,t) = u(\epsilon x, \epsilon^2 t)$: $\mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^L$ satisfies (7.34) with $\epsilon = 1$, we assume $\epsilon = 1$. Using the Ricci identity, we compute

$$\begin{aligned}\left(\partial_t - \Delta\right)\left(\frac{|\nabla u|^2}{2}\right) &= \nabla\left(\partial_t u - \Delta u\right)\cdot\nabla u - |\nabla^2 u|^2 - \mathrm{Ric}^M(u)(\nabla u, \nabla u) \\ &= \nabla(f(u)))\cdot\nabla u - |\nabla^2 u|^2 - \mathrm{Ric}^M(u)(\nabla u, \nabla u),\end{aligned}$$

and

$$\begin{aligned}\left(\partial_t - \Delta\right)\left(F(u)\right) &= -f(u)\left(\partial_t u - \Delta u\right) + \nabla(f(u))\cdot\nabla u \\ &= -|f(u)|^2 + \nabla(f(u))\cdot\nabla u.\end{aligned}$$

Combining these two identities, we obtain

$$
\begin{aligned}
&(\partial_t - \Delta) e(u) + |\nabla^2 u|^2 + |f(u)|^2 + \mathrm{Ric}^M(u)(\nabla u, \nabla u) \\
&= 2\nabla(f(u)) \cdot \nabla u \\
&= -2\nabla \left(\lambda'(\cdot) \frac{d}{du}(\mathrm{dist}^2(u,N)) \right) \cdot \nabla u \\
&= -2\chi''(\cdot) \left| \frac{d}{du}(\mathrm{dist}^2(u,N) \cdot \nabla u \right|^2 \\
&\quad -2\chi'(\cdot) \frac{d^2}{du^2}(\mathrm{dist}^2(u,N))(\nabla u, \nabla u) \\
&= -4 \left(\chi'(\cdot) + \chi''(\cdot)\mathrm{dist}^2(u,N) \right) \left| \frac{d}{du}(\mathrm{dist}(u,N)) \cdot \nabla u \right|^2 \\
&\quad -4\chi'(\cdot)\mathrm{dist}(u,N) \left(\frac{d^2}{du^2}\mathrm{dist}(u,N)(\nabla u, \nabla u) \right) \\
&\le -4\chi''(\cdot)\mathrm{dist}^2(u,N) \left| \frac{d}{du}(\mathrm{dist}(u,N)) \cdot \nabla u \right|^2 \\
&\quad -4\chi'(\cdot)\mathrm{dist}(u,N) \left(\frac{d^2}{du^2}\mathrm{dist}(u,N)(\nabla u, \nabla u) \right)
\end{aligned}
$$

where we have used the fact that $\chi'(\cdot) \ge 0$ in the last step. If $\mathrm{dist}(u,N) \le 2\delta_N$, then

$$\left| \frac{d}{du} \left(\mathrm{dist}^2(u,N) \right) \right| = 2\mathrm{dist}(u,N),$$

and

$$\left| \frac{d^2}{du^2} \left(\mathrm{dist}(u,N) \right) \right| \le C(N),$$

where $C(N)$ depends only on the bound of the sectional curvature of N. Hence we have

$$\left| -4\chi'(\cdot)\mathrm{dist}(u,N) \left(\frac{d^2}{du^2}\mathrm{dist}(u,N)(\nabla u, \nabla u) \right) \right| \le \frac{1}{2}\chi'(\cdot)^2 \mathrm{dist}^2(u,N) + c|\nabla u|^4,$$

and

$$\left| -2\chi''(\cdot)(\mathrm{dist}^2(u,N)) \left| \frac{d}{du}(\mathrm{dist}^2(u,N)) \cdot \nabla u \right|^2 \right| \le \mathrm{dist}^2(u,N) + c|\nabla u|^4.$$

Note also that

$$\left| \mathrm{Ric}^M(\nabla u, \nabla u) \right| \le C_M \left| \nabla u \right|^2,$$

where C_M depends on the bound of the Ricci curvature of M. Putting all these estimates together, we obtain (7.39). □

With Lemma 7.6.2 and Lemma 7.6.3, we can obtain the following gradient estimate under the smallness of renormalized energy. The proof is exactly same as that of Proposition 7.1.4.

Lemma 7.6.4 *Suppose that $u_\epsilon \in C^\infty(M \times \mathbb{R}_+, \mathbb{R}^L)$ solves (7.34). Then there exist positive constants $\epsilon_0 > 0$, δ_0 and $c > 0$ depending only on M and N such that if for $z_0 = (x_0, t_0) \in M \times (0, +\infty)$, there is $0 < R < \min\{\sqrt{t_0}, i_M\}$ such that*

$$\Psi(u_\epsilon, z_0, R) \le \epsilon_0^2$$

then

$$\sup_{P_{\delta_0 R}(z_0)} e_\epsilon(u_\epsilon) \le c(\delta_0 R)^{-2}. \tag{7.40}$$

Now we prove the main theorem of this section (see [33]).

Theorem 7.6.5 *For any $u_0 \in C^\infty(M, N)$ there exists a global weak solution $u : M \times \mathbb{R}_+ \to N$ of (5.3) and (5.4), which is smooth away from a singular set Σ of locally finite n-dimensional parabolic Hausdorff measure. Moreover, $\partial_t u \in L^2(M \times \mathbb{R}_+)$ and $E(u(t)) \le E(u_0)$ for a.e. t, and $\Sigma(t) = \Sigma \cap \{t\}$ has locally finite $(n-2)$-dimensional Hausdorff measure for all $t > 0$. Finally, there exists $t_i \uparrow +\infty$ such that $u(t_i)$ converges weakly in $H^1(M, N)$ to a harmonic map $u_\infty : M \to N$, with $E(u_\infty) \le E(u_0)$ and $u_\infty \in C^\infty(M \setminus \Sigma_\infty, N)$, where $\Sigma_\infty \subset M$ is a closed set with $H^{n-2}(\Sigma_\infty) \le C(E(u_0))$.*

Proof. For $\epsilon > 0$, let $u_\epsilon \in C^\infty(M \times \mathbb{R}_+, \mathbb{R}^L)$ solve (7.34) with initial data $u_\epsilon(0) = u_0$. By Lemma 7.5.2 we can assume, after taking possible subsequences, that there is $u : M \times \mathbb{R}_+ \to N$ with $\partial_t u \in L^2(M \times \mathbb{R}_+)$ and $\nabla u \in L^\infty(\mathbb{R}_+, L^2(M))$ such that

$$\nabla u_\epsilon \to \nabla u \text{ weak}^* \text{ in } \; L^\infty(\mathbb{R}_+, L^2(M)),$$

$$\partial_t u_\epsilon \to \partial_t u \text{ weakly } \text{ in } \; L^2(M \times \mathbb{R}_+),$$

$$u_\epsilon \to u \text{ in } L^2_{\text{loc}}(M \times \mathbb{R}_+).$$

By the lower semicontinuity and (7.6.1), we have for a.e. $t \in \mathbb{R}_+$,

$$E(u(t)) \le \liminf_{\epsilon \to 0} E(u_\epsilon(t)) \le \liminf_{\epsilon \to 0} E_\epsilon(u_\epsilon(t)) \le E(u_0).$$

Define

$$\Sigma = \bigcap_{R>0} \left\{ z_0 = (x_0, t_0) \in M \times \mathbb{R}_+ \mid \liminf_{\epsilon > 0} \Psi(u_\epsilon z_0, R) \ge \epsilon_0^2 \right\} \tag{7.41}$$

where ϵ_0 is given by Lemma 7.6.4.

The proof of that Σ is closed and has locally finite n-dimensional parabolic Hausdorff measure is as same as that of Theorem 7.2.3, except that we replace the density $\frac{1}{2}|\nabla u_k|^2$ by $e_\epsilon(u_\epsilon)$.

To see that $u \in C^\infty(M \times \mathbb{R}_+ \setminus \Sigma, N)$ solves (5.3), we proceed as follows. For $z_0 \notin \Sigma$, we may assume that there exists $0 < R_0 < \min\{\sqrt{t_0}, i_M\}$ such that

$$\Psi(u_\epsilon, z_0, R_0) \le \epsilon_0^2, \quad \text{for all sufficiently small } \epsilon > 0.$$

By Lemma 7.6.4, we have

$$|\nabla u_\epsilon|^2 + \frac{\mathrm{dist}^2(u_\epsilon, N)}{\epsilon^2} \le C(\epsilon_0) \ \text{ on } P_{\delta_0 R_0}(z_0). \tag{7.42}$$

We claim that $[u_\epsilon]_{C^{\frac{1}{2}}(P_{\delta R_0}(z_0))} \le C(\epsilon_0)$. In fact, let $z = (x,t), w = (y,s) \in P_{\frac{R_0}{8}}(z_0)$, and denote $r = \max\{|x-y|, \sqrt{|t-s|}\}$. Then

$$\begin{aligned}
|u_\epsilon(z) - u_\epsilon(w)| &\le \left|u_\epsilon(z) - (u_\epsilon(t))_{B_{2r}(x)}\right| + \left|u_\epsilon(w) - (u_\epsilon(s))_{B_{2r}(x)}\right| \\
&\quad + \left|(u_\epsilon(t))_{B_{2r}(x)} - (u_\epsilon(s))_{B_{2r}(x)}\right| \\
&\le 2\left(\max_{P_{\frac{R_0}{2}}(z_0)} |\nabla u_\epsilon|\right) r + r^{-\frac{n}{2}} \left(\int_s^t \int_{B_{2r}(x)} |\partial_t u_\epsilon|^2\right)^{\frac{1}{2}} |t-s|^{\frac{1}{2}} \\
&\le C(\epsilon_0) r + r^{1-\frac{n}{2}} \left(\int_{P_{2r}(x,t)} |\partial_t u_\epsilon|^2\right)^{\frac{1}{2}} \\
&\le C(\epsilon_0) r + r^{-\frac{n}{2}} \left(\int_{P_{4r}(x,t)} e_\epsilon(u_\epsilon)\right)^{\frac{1}{2}} \\
&\le \left(C(\epsilon_0) + \max_{P_{\frac{R_0}{2}}(z_0)} |\nabla u_\epsilon|\right) r \le C(\epsilon_0) r,
\end{aligned}$$

where $(u_\epsilon(t))_{B_{2r}(x)}$ is the average of $u_\epsilon(t)$ on $B_{2r}(x)$, and we have used the inequality

$$\int_{P_R(z_0)} |\partial_t u_\epsilon|^2 \le C R^{-2} \int_{P_{2R}(z_0)} e_\epsilon(u_\epsilon). \tag{7.43}$$

Hence, after passing to another subsequence, we may assume that

$$u_\epsilon \to u \ \text{ in } \ C^0(P_{\delta R_0}(z_0)),\ \nabla u_\epsilon \to \nabla u \text{ weak}^* \ \text{ in } \ L^\infty(P_{\delta R_0}(z_0)). \tag{7.44}$$

Hence $\chi'(\mathrm{dist}^2(u_\epsilon, N)) = 1$ for ϵ sufficiently small so that the proof of Lemma 7.6.3 gives

$$(\partial_t - \Delta)\, e_\epsilon(u_\epsilon) + \frac{\mathrm{dist}^2\,(u_\epsilon, N)}{\epsilon^2} \le C \tag{7.45}$$

in the sense of distribution on $P_{\delta R_0}(z_0)$, with a constant C independent of ϵ. Choose a nonnegative test function $\phi \in C_0^\infty(P_{\delta R_0}(z_0))$ and multiply (7.45) by ϕ and integrate over $P_{\delta R_0}(z_0)$. Then after integration by parts we have

$$\int_{P_{\delta R_0}(z_0)} \frac{\mathrm{dist}^2(u_\epsilon, N)}{\epsilon^2} \phi \le \int_{P_{\delta R_0}(z_0)} |\partial_t \phi + \Delta\phi|\, e_\epsilon(u_\epsilon) + C\phi \le C(\phi)$$

uniformly in ϵ. It follows that $\frac{\mathrm{dist}(u_\epsilon, N)}{\epsilon}$ is uniformly bounded in $L^2_{\mathrm{loc}}(P_{\delta R_0}(z_0))$. Hence $(\partial_t - \Delta) u_\epsilon$ is uniformly bounded in $L^2_{\mathrm{loc}}(P_{\delta R_0}(z_0))$ and $\partial_t u_\epsilon, \nabla^2 u_\epsilon$ are also uniformly bounded in $L^2_{\mathrm{loc}}(P_{\delta R_0}(z_0))$. Therefore

$$(\partial_t - \Delta)\, u_\epsilon \to (\partial_t - \Delta)\, u \ \text{ weakly in } \ L^2_{\mathrm{loc}}(P_{\delta R_0}(z_0)), \tag{7.46}$$

and for some $\nu \in L^2_{\text{loc}}(P_{\delta R_0}(z_0))$,

$$\frac{\text{dist}(u_\epsilon, N)}{\epsilon} \to \nu \;\text{ weakly in }\; L^2_{\text{loc}}(P_{\delta R_0}(z_0)). \tag{7.47}$$

Note also that there exists a unit vector field $\nu_N^{(\epsilon)} \perp T_{\pi_N(u_\epsilon)}N$ such that

$$\frac{d}{du}\left(\text{dist}^2(u_\epsilon, N)\right) = 2\text{dist}(u_\epsilon, N)\nu_N^{(\epsilon)}.$$

Therefore we have

$$\lim_{\epsilon\to 0}\int_{P_{\delta R_0}(z_0)} \frac{1}{\epsilon^2}\frac{d}{du}\left(\text{dist}^2(u_\epsilon, N)\right)\cdot \phi\, dv_g dt = 0, \tag{7.48}$$

for any vector field $\phi \in L^2_{\text{loc}}(P_{\delta R_0}(z_0))$ with $\phi(z) \in T_{u(z)}N$ for a.e. $z \in P_{\delta R_0}(z_0)$. Thus we have $(\partial_t - \Delta)u \perp T_uN$ a.e. in $P_{\delta R_0}(z_0)$ so that there exists a unit normal vector field $\nu_N(u)$ of N along u and a multiplier function $a \in L^2_{\text{loc}}(P_{\delta R_0}(z_0))$ such that

$$\partial_t u - \Delta u = a\nu_N(u) \;\text{ a.e. }\; P_{\delta R_0}(z_0). \tag{7.49}$$

This is equivalent to the geometric form of (5.3):

$$\partial_t u - \Delta u \perp T_uN.$$

Moreover, since $u \in C^0(P_{\delta R_0}(z_0), N)$, the higher order regularity theory implies that $u \in C^\infty(P_{\delta R_0}(z_0), N)$. Hence $u \in C^\infty(M \times \mathbb{R}_+ \setminus \Sigma, N)$ solves (5.3).

The proof of that u weakly solves (5.3) in $M \times \mathbb{R}_+$ can be done in the same way as that of Theorem 7.2.3. The refined estimate of Σ_t for $t \in \mathbb{R}_+$ is due to [26].

For the last part of this theorem, we first choose a sequence $t_\epsilon \to \infty$ such that $u_\epsilon(t_\epsilon) \to u_\infty$ weakly in $H^1(M, N)$, while

$$\int_M |\partial u_\epsilon(t_\epsilon)|^2 \to 0 \;\text{ and }\; \int_{t_\epsilon - 1}^{t_\epsilon}\int_M |\partial_t u_\epsilon|^2\, dv_g dt \to 0, \text{ as } \epsilon \to 0. \tag{7.50}$$

Define

$$\Sigma_\infty = \bigcap_{R>0}\bigcup_{0<r<R}\left\{x_0 \in M \mid \liminf_{\epsilon\to 0}\Psi(u_\epsilon, (x_0, t_\epsilon), R) \geq \epsilon_0^2\right\}$$

where ϵ_0 is given by Lemma 7.6.4.

For simplicity, assume Σ_∞ is compact. To estimate $H^{n-2}(\Sigma_\infty)$, let $R > 0$ be sufficiently small and $\{B_{R_i}(x_i)\}_{i\in I}, x_i \in \Sigma,\ R_i \leq R$, be an open cover of Σ_∞. By compactness and Vitali's covering Lemma there is a finite subfamily $I' \subset I$ such that $\{B_{2R_i}(x_i)\}_{i\in I'}$ are mutually disjoint and $\{B_{10R_i}(x_i)\}_{i\in I'}$ covers Σ_∞.

For any $\delta > 0$ there is a constant $C(\delta) > 0$ such that for any $i \in I'$ there is

$r_i \in (0, \frac{R_i}{C(\delta)})$ such that

$$\begin{aligned}
\epsilon_0^2 &\leq \int_{T_{r_i}(x_i,t_\epsilon)} e_\epsilon(u_\epsilon) G_{(x_i,t_\epsilon)} \phi^2 \, dv_g dt \\
&\leq 4r_i^{-2} \int_{t_\epsilon - r_i^2}^{t_\epsilon - \frac{r_i^2}{4}} \left(r_i^2 \int_{S_{r_i}(x_i,t_\epsilon)} e_\epsilon(u_\epsilon) G_{(x_i,t_\epsilon)} \phi^2 \, dv_g \right) dt \\
&\leq c \exp(cr_i) \, r_i^2 \int_{S_{r_i}(x_i,t_\epsilon)} e_\epsilon(u_\epsilon) G_{(x_i,t_\epsilon)} \phi^2 \, dv_g + cr_i E_0 \\
&\leq c \exp\left(c(\frac{R_i}{C(\delta)} - r_i) \right) \left(\frac{R_i}{C(\delta)} \right)^2 \int_{S_{\frac{R_i}{C(\delta)}}(x_i,t_\epsilon)} e_\epsilon(u_\epsilon) G_{(x_i,t_\epsilon)} \phi^2 \, dv_g + cR_i E_0 \\
&\leq C(\delta) R_i^{2-n} \int_{B_{R_i}(x_i) \times \{t_\epsilon - (\frac{R_i}{C(\delta)})^2\}} e_\epsilon(u_\epsilon) \, dv_g + (\delta + CR) \, E_0 \qquad (7.51)
\end{aligned}$$

where we have applied (7.37) and

$$G_{(x_i,t_\epsilon)} \leq \delta \ \text{ on } \ \left(S_{\frac{R_i}{C(\delta)}}(x_i,t_\epsilon) \setminus B_{R_i}(x_i) \right) \times \left\{ t_\epsilon - (\frac{R_i}{C(\delta)})^2 \right\}.$$

By choosing $\delta = \frac{\epsilon_0^2}{3E_0}$ and $R < \frac{\epsilon_0^2}{3C_0 E_0}$, we then have that for any $i \in I'$ and ϵ sufficiently small,

$$\epsilon_0^2 R_i^{n-2} \leq C \int_{B_{R_i}(x_i) \times \{t_\epsilon - (\frac{R_i}{C(\delta)})^2\}} e_\epsilon(u_\epsilon) \, dv_g.$$

We now need another local energy inequality.

Lemma 7.6.6 *Suppose that $u_\epsilon \in C^\infty(M \times \mathbb{R}_+, \mathbb{R}^L)$ solves (7.34). For $x_0 \in M$, let $0 < R < \frac{i_M}{2}$ and $0 \leq S < T < +\infty$ be given. Then*

$$\begin{aligned}
\int_{B_{2R}(x_0) \times \{T\}} e_\epsilon(u_\epsilon) &\geq c \int_{B_R(x_0) \times \{S\}} e_\epsilon(u_\epsilon) - \int_S^T \int_M |\partial_t u_\epsilon|^2 \\
&\quad - c \left(\frac{T-S}{R^2} E_0 \int_S^T \int_M |\partial_t u_\epsilon|^2 \right)^{\frac{1}{2}} \qquad (7.52)
\end{aligned}$$

where $E_0 = E(u_0)$ and c depends only on M and N.

Proof. Let $\phi \in C_0^\infty(B_{2R}(x_0))$ with $\phi \equiv 1$ on $B_R(x_0)$ and $|\nabla \phi| \leq CR^{-1}$. Multiplying (7.34) by $\phi^2 \partial_t u_\epsilon$ and integrating over $M \times [S,T]$, we obtain

$$\begin{aligned}
&\int_S^T \frac{d}{dt} \left(\int_M e_\epsilon(u_\epsilon) \phi^2 \right) + \int_S^T \int_M |\partial_t u_\epsilon|^2 \, \phi^2 \\
&\geq - \int_S^T \int_M |\nabla u_\epsilon| \, |\nabla \phi| \, |\partial_t u_\epsilon| \, \phi \\
&\geq - \frac{C}{R} \left(\int_S^T \int_M |\partial_t u_\epsilon|^2 \phi^2 \right)^{\frac{1}{2}} \left(\int_S^T \int_M |\nabla u_\epsilon|^2 \right)^{\frac{1}{2}}.
\end{aligned}$$

(7.52) now follows by using the estimate

$$\int_S^T \int_M |\nabla u_\epsilon|^2 \le (T-S) \sup_{S\le t\le T} E_\epsilon(u_\epsilon) \le (T-S)E_0.$$

□

Applying (7.52) with $R = R_i$, $S = t_\epsilon - (\frac{R_i}{C(\delta)})^2$ and $T = t_\epsilon$, we obtain that for any $i \in I'$ and ϵ sufficiently small,

$$\begin{aligned}\epsilon_0^2 R_i^{n-2} &\le C[\int_{B_{2R_i}(x_i)\times\{t_\epsilon\}} e_\epsilon(u_\epsilon) \\ &\quad + \left(1 + \|\partial_t u_\epsilon\|_{L^2(M\times[t_\epsilon-1,t_\epsilon])}\right) \|\partial_t u_\epsilon\|_{L^2(M\times[t_\epsilon-1,t_\epsilon])}] \\ &\le C \int_{B_{2R_i}(x_i)\times\{t_\epsilon\}} e_\epsilon(u_\epsilon) + o(1).\end{aligned}$$

Therefore we have

$$\begin{aligned}H^{n-2}_{10R}(\Sigma_\infty) &\le \sum_{i\in I'} (10R_i)^{n-2} = 10^{n-2} \sum_{i\in I'} R_i^{n-2} \\ &\le C\epsilon_0^{-2} \left(\int_{\cup_{i\in I'} B_{2R_i}(x_i)} e_\epsilon(u_\epsilon(t_\epsilon) \right) + |I'|o(1) \\ &\le C(\epsilon_0)E_\epsilon(u_\epsilon(t_\epsilon)) + o(1) \le CE_0 + o(1).\end{aligned}$$

Taking R and ϵ into zero, we obtain $H^{n-2}(\Sigma_\infty) \le C(E_0)$.

For $x_0 \notin \Sigma_\infty$, there exists $R_0 > 0$ such that for subsequences $\epsilon_k \to 0$ of ϵ and $t_k = t_{\epsilon_k} \to \infty$ we have

$$\int_{T_{r_0}(x_0,t_k)} e_{\epsilon_k}(u_{\epsilon_k}) G_{(x_0,t_k)} \phi^2 \, dv_g \le \epsilon_0^2,$$

Hence, by Lemma 7.6.4 we have

$$|\nabla u_{\epsilon_k}|, \ \frac{\mathrm{dist}^2(u_{\epsilon_k}, N)}{\epsilon_k^2} \le C \ \text{ uniformly on } P_{\delta_0 R_0}(x_0, t_k).$$

Moreover, since

$$\partial_t u_{\epsilon_k}(t_k) \to 0 \ \text{ in } \ L^2(M),$$

we have $u_{\epsilon_k} \to u_\infty$ uniformly on $B_{\delta_0 R_0}(x_0)$ and weakly in $W^{2,2}(B_{\delta_0 R_0}(x_0), N)$. Therefore $u_\infty \in C^\infty(M \setminus \Sigma_\infty, N)$ is a smooth harmonic map. Since $u_\infty \in H^1(M,N)$ and $H^{n-2}(\Sigma_\infty) < +\infty$, u_∞ is also a weakly harmonic map on M. □

Remark 7.6.7 Theorem 7.2.3 remains to be true for $M = \mathbb{R}^n$. In this case, u_∞ is constant for $n \ge 3$.

Proof. Note that by (7.7) we have that for $t_0 > 0$ and $R_0 = \frac{\sqrt{t_0}}{2}$,

$$\int_{T_{R_0}(z_0)} e_\epsilon(u_\epsilon)G_{z_0} \le \int_0^{\frac{3t_0}{4}} \int_{\mathbb{R}^n} e_\epsilon(u_\epsilon)G_{z_0} \le Ct_0^{\frac{2-n}{2}} E_0 \le \epsilon_0^2, \tag{7.53}$$

uniformly in ϵ, if $t_0 \ge C(\frac{E_0}{\epsilon_0^2})^{\frac{2}{n-2}}$. Therefore we have

$$|\nabla u_\epsilon|^2(x,t) \le \frac{C}{t} \tag{7.54}$$

for large t and hence $u(t) \to u_\infty \equiv$ constant as $t \to \infty$. □

When $\partial M \ne \emptyset$, we can consider the Dirichlet boundary value problem of the heat flow of harmonic maps $u : \overline{M} \times \mathbb{R}_+ \to N$:

$$\partial_t u - \Delta u = A(u)(\nabla u, \nabla u), \ (x,t) \in M \times \mathbb{R}_+ \tag{7.55}$$

$$u(x,0) = u_0(x), \quad x \in M \tag{7.56}$$

$$u(x,t) = u_0(x), \quad (x,t) \in \partial M \times \mathbb{R}_+, \tag{7.57}$$

where $u_0 : \overline{M} \to N$ is given. Without any curvature assumption on N, Chen-Lin [31] have extended the main result by Chen-Struwe [33] and obtained the following theorem.

Theorem 7.6.8 *There is a global weak solution $u : \overline{M} \times \mathbb{R}_+ \to N$ to (7.55)-(7.57) with $\partial_t u \in L^2(M \times \mathbb{R}_+)$ and $\nabla u \in L^\infty(\mathbb{R}_+, L^2(M))$ such that $u \in C^\infty(\overline{M} \times \mathbb{R}_+ \setminus \Sigma, N)$, where $\Sigma \subset \overline{M} \times \mathbb{R}_+$ is a closed subset which has a locally finite n-dimensional parabolic Hausdorff measure. Moreover, as t $\to +\infty$ suitably, $u(t)$ converges weakly in $H^1(M,N)$ to a weakly harmonic map $u_\infty : \overline{M} \to N$ with $u_\infty|_{\partial M} = u_0|_{\partial M}$, which is smooth away from a closed $\Sigma_\infty \subset \overline{M}$ with $H^{n-2}(\Sigma_\infty)$ bounced by $\|u_0\|_{C^2(\partial M)}$ and $E(u_0)$.*

Remark 7.6.9 When $\partial\Omega \ne \emptyset$, we can also consider the free boundary value problem of the heat flow of harmonic maps, namely (7.55), (7.57), and (7.56) replaced by $\frac{\partial u}{\partial \nu}(x,t) = 0$ for $x \in \partial M$ and $t > 0$. We refer the readers to Ma [141], Struwe [197], and Chen-Lin [32] for some interesting works in this direction.

Chapter 8

Blow up analysis on heat flows

From both Chapter 2 and Chapter 4, we see that the main obstruction to smoothness of harmonic maps from (M,g) to (N,h) is the existence of *nontrivial* smooth harmonic maps from S^k to N, called harmonic S^k's, $2 \le k \le n-1$ ($n = \dim M$). More precisely, it is proved in Chapter 2 that if (N,h) does not admit harmonic S^k's for $2 \le k \le n-1$, then any energy minimizing harmonic map $u \in H^1(M,N)$ is smooth and satisfies the following estimate

$$\|u\|_{C^l(M)} \le C(l, M, N, E(u_0)). \tag{8.1}$$

The above theorem was then proved by Lin [123] for the class of *stationary* harmonic maps (see Chapter 4). By the maximum principle, it is well known that any (N,h) does not support any harmonic S^k for $2 \le k \le n-1$, if either (N,h) has nonpositive sectional curvature and $\pi_1(N) = 0$ or supports a strictly convex function. Therefore, the above theorem extends the classical theorems by Eells-Sampson [49] (see also Chapter 5) (in the steady case) and by Hildebrandt-Kaul-Widman [95] (see also §3.1).

In this chapter, we consider the heat flow of harmonic maps to general Riemannian manifolds:

$$\partial_t u = \Delta_g u + A(u)(\nabla u, \nabla u) \ \text{ in } M \times (0, +\infty), \tag{8.2}$$

$$u|_{t=0} = \phi, \ \text{ on } M, \tag{8.3}$$

where $\phi \in C^\infty(M,N)$ is a given map. It is well known (see Chapter 5) that if $K^N \le 0$ then (8.2) and (8.3) has a unique smooth solution $u \in C^\infty(M \times \mathbb{R}_+, N)$, which satisfies

$$\sup_{t \ge t_0} \|\nabla u(t)\|_{C^0(M)} \le C(t_0, E(\phi)), \ \forall t_0 > 0. \tag{8.4}$$

For a short time smooth solution u to (8.2), there exist $t_0 > 0$ and $C_0 > 0$ depending on M, N and $\|\phi\|_{C^2(M)}$ such that

$$\|\nabla u(t)\|_{C^0(M)} \le C_0, \ \text{ for } 0 \le t < t_0. \tag{8.5}$$

Note that the validity of (8.5) is local, since u may develop finite time singularity (see §7.3). Motivated by the corresponding study on harmonic maps, a natural question is to find all possible obstructions to the existence of global smooth solutions to (8.2) satisfying (8.4) or the global validity of (8.5).

In this chapter, we will present some of the main theorems by Lin and Wang [134], in which we have obtained both necessary and sufficient conditions on (N, h), other than the curvature condition of N or the image of u, such that (8.4) holds for u.

The organization of this chapter is as follows. In §8.1, we state the main theorems of this chapter. In §8.2, we provide some basic estimates on the heat flow of harmonic maps or the Ginzburg-Landau heat flow established in §7. In §8.3, we present a stratification result for the heat flow of harmonic maps by Lin-Wang [136], analogous to that of the singular set of minimizing harmonic maps in §2.3 and stationary harmonic maps in §4.2. In §8.4, we perform the blow-up analysis and prove the main theorems in dimensions two. In §8.5, we perform the blow-up analysis in higher dimensions and complete the proof of the main theorems.

8.1 Obstruction to strong convergence

From the discussion in §6, it is clear that the existence of harmonic S^2's in (N, h) acts as the sole obstruction to the regularity for the heat flow of harmonic maps from Riemannian surfaces.

As a matter of fact, Lin-Wang [134] have proved that harmonic S^2 is the obstruction to strong convergence of sequences of weakly convergent heat flows of harmonic maps in dimensions at least two. More precisely, we have

Theorem 8.1.1 *Assume (N, h) does not admit any harmonic S^2. For $0 < T < +\infty$, suppose that $\{u_i\} \subset C^\infty(M \times [0,T], N)$ satisfies (8.2), with*

$$\sup_i \int_{M\times[0,T]} |\partial_t u_i|^2 + |\nabla u_i|^2 \equiv K < +\infty \tag{8.6}$$

If $u_i \to u$ weakly in $H^1(M \times [0,T], N)$, then $u_i \to u$ strongly in $H^1_{loc}(M \times [0,T], N)$. Hence u is a suitable weak solution of (8.2) with $\mathcal{P}^n(sing(u)) = 0$. Moreover, u satisfies both the energy equality: for any $\phi \in C_0^\infty(M)$ and a.e. $t > 0$,

$$\frac{d}{dt}\int_M |\nabla u|^2\phi^2 = -2\int_M \left(|\partial_t u|^2\,\phi^2 + \langle \nabla u, \partial_t u\rangle \nabla\phi^2\right), \tag{8.7}$$

and the energy monotonicity formula (7.4) and (7.7).

It turns out that the idea to approach Theorem 8.1.1 is flexible enough to cover the issue of convergence for the Ginzburg-Landau (approximate) heat flows. More precisely, using the same notations as §7.6, we are considering that $u_\epsilon \in C^\infty(M \times [0,T), \mathbb{R}^L)$, $\epsilon > 0$, solves

$$\partial_t u_\epsilon - \Delta u_\epsilon - \frac{1}{\epsilon^2} f(u_\epsilon) = 0, \text{ in } M \times (0,T), \tag{8.8}$$

with the same initial condition (8.3). Let

$$e_\epsilon(u_\epsilon) = \left(\frac{1}{2}|\nabla u_\epsilon|^2 + \frac{F(u_\epsilon)}{\epsilon^2}\right)$$

be the Ginzburg-Landau energy density function. Then we have

Theorem 8.1.2 *Assume that (N, h) does not admit any harmonic S^2. For $0 < T < +\infty$ fixed, let $\epsilon_i \downarrow 0$ and $u_{\epsilon_i} \in C^\infty(M \times [0,T), \mathbb{R}^L)$ be a sequence of solutions to (8.8) with*

$$\sup_i \int_{M\times[0,T]} \left(|\partial_t u_{\epsilon_i}|^2 + e_{\epsilon_i}(u_{\epsilon_i})\right) = K < +\infty. \tag{8.9}$$

If $u_{\epsilon_i} \to u$ weakly in $H^1(M \times (0,T), \mathbb{R}^L)$. Then

$$e_{\epsilon_i}(u_{\epsilon_i})\, dx\, dt \rightharpoonup \frac{1}{2}|\nabla u|^2\, dxdt, \tag{8.10}$$

as convergence of Radon measures on $M \times [0,T]$ and hence $u_{\epsilon_i} \to u$ strongly in $H^1_{loc}(M \times [0,T], \mathbb{R}^L)$. Thus u is a suitable weak solution of (8.2), with $\mathcal{P}^n(\mathrm{sing}(u)) = 0$, and satisfies (8.7) and (7.4) and (7.7).

Once we have the strong convergence, we can employ a parabolic version of Federer dimension reduction argument to find all possible obstruction to C^1 convergence. In order to state the result, we introduce the definition of *quasi-harmonic spheres.*

Definition 8.1.3 For $l \geq 3$, a map $\psi \in C^\infty(\mathbb{R}^l, N)$ is called quasi-harmonic S^l, if ψ is a smooth map from $\mathbb{R}^l$ to N such that (i)

$$0 < E_q(\psi) \equiv \int_{\mathbb{R}^l} |\nabla\psi|^2 e^{-\frac{|y|^2}{4}}\, dy < +\infty,$$

(ii) ψ is a critical point of E_q, i.e., ψ satisfies the quasi harmonic sphere equation:

$$\Delta\psi + A(\psi)(\nabla\psi, \nabla\psi) = \frac{1}{2} y \cdot \nabla\psi,\ y \in \mathbb{R}^l. \tag{8.11}$$

Quasi-harmonic spheres are closely related to nontrivial, self-similar solutions to (8.2). In fact, we have

Remark 8.1.4 $\psi : \mathbb{R}^l \to N$ is a quasi harmonic S^l, $l \geq 3$, iff $v(x,t) = \psi(\frac{x}{\sqrt{-t}}) \in C^\infty(\mathbb{R}^l \times (-\infty, 0), N)$ is a solution of (8.2), and

$$0 < |t| \int_{\mathbb{R}^l} |\nabla v|^2(x,t) G_{(0,0)}(x,t)\, dx = \frac{1}{(4\pi)^{\frac{n}{2}}} E_q(\psi) < +\infty,\ \forall t < 0.$$

It is not hard to check that for any Riemannian manifold N, there doesn't exist quasi-harmonic S^2 in N.

For a sequence of heat flows of harmonic maps, we have

Corollary 8.1.5 *Assume that (N, h) supports neither harmonic S^k for $2 \leq k \leq n-1$ nor quasi-harmonic S^l for $3 \leq l \leq n$. Suppose that $\{u_i\} \subset C^\infty(M \times (0,T), N)$ is a sequence of solutions of (8.2) and $u_i \to u$ weakly in $H^1(M \times [0,T], N)$. Then $u_i \to u$ in $C^2_{loc}(M \times (0,T], N)$ and $u \in C^\infty(M \times (0,T], N)$ is a smooth solution of (8.2) satisfying (8.5) with $t_0 = T$.*

For a sequence of solutions to the Ginzburg-Landau heat flow, we have

Corollary 8.1.6 *Assume that (N,h) admits neither harmonic S^k for $2 \le k \le n-1$ nor quasi-harmonic S^l for $3 \le l \le n$. For $\epsilon_i \downarrow 0$, let $u_{\epsilon_i} \in C^\infty(M \times (0,2), \mathbb{R}^L)$ solve (8.8) such that $u_{\epsilon_i} \to u$ weakly in $H^1(M \times [0,T], \mathbb{R}^L)$. Then u_{ϵ_i} converges to u in $C^2_{loc}(M \times (0,T], \mathbb{R}^L)$. In particular, $u \in C^\infty(M \times (0,T], N)$ solves (8.2) and satisfies (8.5) with $t_0 = T$.*

As another consequence, we can prove

Theorem 8.1.7 *For any $\phi \in C^\infty(M,N)$, either (8.2)-(8.3) has a unique smooth solution $u \in C^\infty(M \times \mathbb{R}_+, N)$ which satisfies (8.4), or (N,h) admits either harmonic S^k for some $2 \le k \le n-1$ or quasi-harmonic S^l for some $3 \le l \le n$.*

8.2 Basic estimates

In thise section, we collect some basic estimates and preliminary facts for (8.2) and (8.8), most of which have been established in §7.
To simplify our presentation, we choose to collect these facts on (8.8) and leave the readers for those ones for (8.2).

For $\epsilon > 0$, let $u_\epsilon \in C^\infty(M \times \mathbb{R}_+, \mathbb{R}^L)$ solve (8.8). Write $e(u_\epsilon)$ for $e_\epsilon(u_\epsilon)$. We begin with the energy equality for u_ϵ.

Lemma 8.2.1 *For any $\phi \in C^1_0(M, \mathbb{R}_+)$ and $0 \le t_1 \le t_2 \le \infty$,*

$$\begin{aligned} &\int_M e(u_\epsilon(t_1))\phi - \int_M e(u_\epsilon(t_2))\phi \\ = &\int_{t_1}^{t_2} \int_M \phi(x) \left|\partial_t u_\epsilon\right|^2 + \int_{t_1}^{t_2} \int_M \nabla\phi(x) \cdot \langle \nabla u_\epsilon \cdot \partial_t u_\epsilon \rangle . \end{aligned} \tag{8.12}$$

In particular,

$$\int_M e(u_\epsilon(t_1)) - \int_M e(u_\epsilon(t_2)) = \int_{t_1}^{t_2} \int_M |\partial_t u_\epsilon|^2 . \tag{8.13}$$

We next recall the energy monotonicity inequality for u_ϵ.

Lemma 8.2.2 *Let $u_\epsilon \in C^\infty(M \times \mathbb{R}_+, \mathbb{R}^L)$ solve (8.8). Then*

$$\begin{aligned} &c \int_{R_1}^{R_2} \frac{1}{r} \int_{T_r(z_0)} \left(\eta^2 \frac{|(x-x_0)\cdot \nabla u_\epsilon + 2(t-t_0)\partial_t u_\epsilon|^2}{|t_0 - t|} + \eta^2 \frac{F(u_\epsilon)}{\epsilon^2} \right) G_{z_0} \\ &\le e^{C(R_2 - R_1)} \Psi\left(u_\epsilon, z_0, R_2\right) - \Psi\left(u_\epsilon, z_0, R_1\right) + C E_0 \left(R_2 - R_1\right) \end{aligned} \tag{8.14}$$

and

$$\Phi\left(u_\epsilon, z_0, R_1\right) \le e^{C(R_2 - R_1)} \Phi\left(u_\epsilon, z_0, R_2\right) + C E_0 \left(R_2 - R_1\right) \tag{8.15}$$

for $z_0 \in M \times \mathbb{R}_+$ and $0 < R_1 \le R_2 < \min\{\frac{\sqrt{t_0}}{2}, i_M\}$. Here $c, C > 0$ depend only on M, m, N, and $E_0 = \frac{1}{2}\int_M |\nabla u_0|^2$.

We also recall the ϵ_0-apriori estimate for u_ϵ.

Lemma 8.2.3 *There exist* $\epsilon_0, \delta_0, C_0 > 0$ *such that if for* $0 < R \le \min\{\frac{\sqrt{t_0}}{2}, i_M\}$, $\Psi(u_\epsilon, z_0, R) \le \epsilon_0^2$, *then*

$$\sup_{z \in P_{\delta_0 R}(z_0)} e(u_\epsilon)(z) \le C_0(\delta_0 R)^{-2}. \tag{8.16}$$

For $\epsilon_i \downarrow 0$, assume that $u_i \equiv u_{\epsilon_i} \to u$ weakly in $H^1_{\text{loc}}(M \times \mathbb{R}_+, \mathbb{R}^L)$. Then there exist two nonnegative Radon measures ν, η on $M \times \mathbb{R}_+$ such that

$$e(u_i)(x,t)\,dxdt \to \frac{1}{2}|\nabla u|^2(x,t)\,dxdt + \nu \equiv \mu,$$

$$|\partial_t u_i|^2(x,t)\,dxdt \to |\partial_t u|^2(x,t)\,dxdt + \eta,$$

as convergence of Radon measures on $M \times \mathbb{R}_+$. As in §7.6, define the concentration set

$$\Sigma = \bigcup_{0<R<r_0} \left\{ z \in M \times \mathbb{R}_+ : \lim_{i\to\infty} \int_{T_R(z)} \eta^2(x) e(u_i) G_z \ge \epsilon_0^2 \right\} \tag{8.17}$$

with $\epsilon_0 > 0$ given by Lemma 8.2.3. Then by summarizing the main theorems from §7.6, we have

Lemma 8.2.4 *(i)* Σ *is closed and* $\mathcal{P}^n(\Sigma \cap P_R) \le C_R < \infty$ *for any* $R < \infty$.
(ii) for any $t > 0$, $\Sigma_t = \Sigma \cap \{t\}$ *has* $H^{n-2}(\Sigma_t \cap K) \le C_K < \infty$, *for any compact* $K \subset M$.
(iii) $u \in C^\infty(M \times \mathbb{R}_+ \setminus \Sigma, N)$ *is a weak solution of (8.2) on* $M \times \mathbb{R}_+$.
(iv) $u_i \to u$ *strongly in* $H^1_{loc} \cap C^1_{loc}(M \times \mathbb{R}_+ \setminus \Sigma, \mathbb{R}^L)$.

Moreover, we have

Lemma 8.2.5 *For any* $z \in M \times (0, +\infty)$, $\int_{T_R(z)} \eta^2(x) G_z(x,t)\,d\mu(x,t)$ *is monotonically increasing with respect to* R. *Hence*

$$\Theta^n(\mu, z) = \lim_{R\downarrow 0} \int_{T_R(z)} \eta^2(x) G_z(x,t)\,d\mu(x,t)$$

exists and is upper-semicontinuous for all $z \in M \times \mathbb{R}_+$. *In particular,*

$$\Sigma = \left\{ z \in M \times \mathbb{R}_+ \mid \epsilon_0^2 \le \Theta^n(\mu, z) < \infty \right\}. \tag{8.18}$$

Proof. Taking i to ∞ in the monotonicity inequality (8.14) of u_i, we have that for $0 < R \le R_0$,

$$\int_{T_R(z)} \eta^2 G_z\,d\mu \le \int_{T_{R_0}(z)} \eta^2 G_z\,d\mu + CE_0(R_0 - R),$$

this implies $\Theta^n(\mu, z)$ exists and is upper semicontinuous for all $z \in M \times (0, +\infty)$. It is easy to see that (8.17) then implies (8.18). □

Lemma 8.2.6 *For* $\mathcal{P}^n$ *a.e.* $z \in \Sigma$,

$$\Theta^n(u, z) \equiv \lim_{R\downarrow 0} R^{-n} \int_{P_R(z)} |\nabla u|^2(x,t)\,dxdt = 0 \quad \textit{and } \Theta^n(\nu, z) = \Theta^n(\mu, z). \tag{8.19}$$

Proof. The first fact is a consequence of the well-known fact by Federer-Ziemer [56]. The second fact follows from the first fact and Lemma 8.2.5. □

Lemma 8.2.7 *(i) $\Sigma = sing(u) \bigcup supp(\nu)$ and $supp(\eta) \subset \Sigma$.*
(ii) u_i doesn't converge to u strongly in $H^1_{loc}(M \times \mathbb{R}_+, \mathbb{R}^L)$ iff $\mathcal{P}^n(\Sigma) > 0$ and $\nu(M \times \mathbb{R}_+) > 0$.

Proof. First we want to show

$$e(u_i)\, dxdt \to \frac{1}{2}|\nabla u|^2\, dxdt$$

as convergence of Radon measures on $M \times (0, +\infty) \setminus \Sigma$. For this, it suffices to prove that $\nu(P_R(z_0)) = 0$ for any $P_R(z_0) \subset\subset M \times (0, +\infty) \setminus \Sigma$.

For $z_0 \notin \Sigma$, let $r_0 > 0$ be such that $P_{r_0}(z_0) \cap \Sigma = \emptyset$. Hence, by Lemma 8.2.2 we have that if $0 < R \leq R_0 \leq \min\{\frac{\sqrt{t_0}}{2}, r_0\}$ then

$$\begin{aligned}
&\Psi(u_i, z_0, R_0) = \Psi(u_i, z_0, R) \\
&+ \int_R^{R_0} \frac{1}{r}\left(\int_{T_r(z_0)} \left[\frac{1}{|s|}|y \cdot \nabla u_i + 2s\partial_s u_i|^2 + \frac{2F(u_i)}{\epsilon_i^2}\right]\eta^2 G_{z_0}\, dyds\right) dr \\
&- \int_R^{R_0} \frac{1}{r}\left(\int_{T_r(z_0)} \langle \nabla u_i, (y \cdot \nabla u_i + 2s\partial_s u_i)\rangle \cdot \nabla\eta^2 G_{z_0}\, dyds\right) dr.
\end{aligned}$$

Taking $\epsilon_i \downarrow 0$ and applying Lemma 8.2.4 (iv) , this implies

$$\begin{aligned}
&\int_{T_{R_0}(z_0)} \eta^2 G_{z_0}\, d\mu = \int_{T_R(z_0)} \eta^2 G_{z_0}\, d\mu \\
&+ \int_R^{R_0} \frac{1}{r}\left(\int_{T_r(z_0)} \frac{|y \cdot \nabla u + 2s\partial_s u|^2}{|s|}\eta^2 G_{z_0}\, dyds + \eta^2 G_{z_0}\, d\nu\right) dr \\
&- \int_R^{R_0} \frac{1}{r}\left(\int_{T_r(z_0)} \langle \nabla u, (y \cdot \nabla u + 2s\partial_s u)\rangle \cdot \nabla\eta^2 G_{z_0}\, dyds\right) dr,
\end{aligned}$$

where we have used the fact that $\frac{F(u_i)}{\epsilon_i^2} \to \nu$ as convergence of Radon measures in $M \times (0, +\infty) \setminus \Sigma$.

On the other hand, since $u \in C^\infty(M \times (0, +\infty) \setminus \Sigma, N)$ satisfies (8.2), we have

$$\begin{aligned}
&\int_{T_{R_0}(z_0)} \eta^2(\frac{1}{2}|\nabla u|^2) G_{z_0} = \int_{T_R(z_0)} \eta^2(\frac{1}{2}|\nabla u|^2) G_{z_0} \\
&- \int_R^{R_0} \frac{1}{r}\left(\int_{T_r(z_0)} \langle \nabla u, (y \cdot \nabla u + 2s\partial_s u)\rangle \cdot \nabla\eta^2 G_{z_0}\, dyds\right) dr \\
&+ \int_R^{R_0} \frac{1}{r}\left(\int_{T_r(z_0)} \frac{|y \cdot \nabla u + 2s\partial_s u|^2}{|s|}\eta^2 G_{z_0}\, dyds\right) dr.
\end{aligned}$$

Subtracting these two identities gives

$$\int_{T_{R_0}(z_0)} \eta^2 G_{z_0}\, d\nu = \int_{T_R(z_0)} \eta^2 G_{z_0}\, d\nu + 2\int_R^{R_0} \frac{1}{r}\int_{T_r(z_0)} \eta^2 G_{z_0}\, d\nu\, dr.$$

This is equivalent to

$$\frac{d}{dr}\left(\int_{T_r(z_0)} \eta^2 G_{z_0}\, d\nu\right) = \frac{2}{r}\int_{T_r(z_0)} \eta^2 G_{z_0}\, d\nu, \quad \text{for } 0 < r \leq R_0. \tag{8.20}$$

This implies

$$\int_{T_r(z_0)} \eta^2 G_{z_0}\, d\nu = (\frac{r}{R})^2 \int_{T_R(z_0)} \eta^2 G_{z_0}\, d\nu.$$

Thus $\nu(P_R(z_0)) = 0$. In particular, we obtain $\mathrm{sing}(u) \cup \mathrm{supp}(\nu) \subset \Sigma$. On the other hand, we have that $\Sigma \subset \mathrm{sing}(u) \cup \mathrm{supp}(\nu)$. To see this, suppose that $z_0 \notin \mathrm{sing}(u) \cup \mathrm{supp}(\nu)$. Then there exists a sufficiently small $\rho > 0$ such that $u \in C^\infty(P_{2\rho}(z_0), N)$ and $\nu(P_\rho(z_0)) = 0$. Hence

$$\rho^{-n}\int_{P_\rho(z_0)} d\mu = \rho^{-n}\left(\int_{P_\rho(z_0)} \frac{1}{2}|\nabla u|^2 + \nu(P_\rho(z_0))\right) \leq \frac{\epsilon_0^2}{2},$$

and $\rho^{-n}\int_{P_\rho(z_0)} e(u_i) \leq \epsilon_0^2$ for any sufficently large i. Hence by the definition of Σ we have that $z_0 \notin \Sigma$. Thus (i) is proven.

For (ii), note that by (8.19) we have that if $\mathcal{P}^n(\Sigma) > 0$ then for $\mathcal{P}^n$ a.e. $z \in \Sigma$,

$$\Theta^n(\nu, z) = \Theta^n(\mu, z) \geq \epsilon_0^2$$

hence $\nu(M \times \mathbb{R}_+) = \nu(\Sigma) > 0$ and $e(u_i)\, dxdt \not\to \frac{1}{2}|\nabla u|^2\, dxdt$. Since by (8.21) we have that $\frac{F(u_i)}{\epsilon_i^2}\, dxdt \to 0$, u_i doesn't converge to u strongly in $H^1(M \times \mathbb{R}_+)$. It is readily seen that the converse also holds. □

Now we need two additional lemmas for our analysis.

Lemma 8.2.8 *Under the same notations as above, it holds*

$$\lim_{i\to\infty}\int_{M\times[t,T]} \frac{F(u_i)}{\epsilon_i^2} = 0, \;\; \forall 0 < t < T < \infty. \tag{8.21}$$

Proof. First note that for any $\beta > 0$

$$\lim_{i\to\infty}\int_{M\times[t,T]\setminus(\Sigma_t^T)_\beta} \frac{F(u_i)}{\epsilon_i^2} = 0$$

where $\Sigma_t^T = \bigcup_{s=t}^T (\Sigma_s \times \{s\})$ and $(\Sigma_t^T)_\beta = \left\{z \in M \times [t,T] : \delta(z, \Sigma_t^T) \leq \beta\right\}$.

It suffices to show that

$$\lim_{i\to\infty}\int_{(\Sigma_t^T)_\beta} \frac{F(u_i)}{\epsilon_i^2} = O(\beta).$$

This can be seen as follows. For any $z_0 \in \Sigma_t^T$, (8.14) gives

$$\Psi\left(\mu, z_0, \frac{\beta}{2}\right) + \lim_{i\to\infty}\int_{\frac{\beta}{2}}^{\beta}\frac{1}{r}\int_{T_r(z_0)}\frac{F(u_i)}{\epsilon_i^2}G_{z_0} \leq e^{C\beta}\Psi\left(\mu, z_0, \beta\right)$$

for sufficiently small $\beta > 0$. Moreover, since $\lim_{\beta\downarrow 0}\Psi(\mu, z_0, \beta)$ exists, we may assume that

$$\left|\Psi(\mu, z_0, \beta) - \Psi(\mu, z_0, \frac{\beta}{2})\right| = O(\beta), \ \forall \beta << 1.$$

Therefore we have

$$\int_{\frac{\beta}{2}}^{\beta}\frac{1}{r}\lim_{i\to\infty}\int_{T_r(z_0)}\frac{F(u_i)}{\epsilon_i^2}G_{z_0} = O(\beta).$$

This, combined with the Fubini's theorem, implies

$$\lim_{i\to\infty}\int_{T_{\overline{\beta}}(z_0)}\frac{F(u_i)}{\epsilon_i^2}G_{z_0} = O(\beta)$$

for some $\overline{\beta} \in (\frac{\beta}{2}, \beta)$. In particular, we have

$$\lim_{i\to\infty}\int_{B_{\frac{\beta}{2}}(x_0)\times[t_0-\beta^2, t_0-\frac{\beta^2}{4}]}\frac{F(u_i)}{\epsilon_i^2} = O(\beta).$$

This, combined with a simple covering argument, implies (8.21). □

Lemma 8.2.9 *There exists a subsequence $i' \to \infty$ such that*

$$e(u_{i'})(x,t)\,dx \to \mu_t, \ \forall t > 0,$$

as convergences of Radon measures, for a family of nonnegative Radon measures $\{\mu_t\}_{t>0}$ on M. In particular, $\mu_t = \frac{1}{2}|\nabla u|^2(x,t)\,dx + \nu_t$, $\mu = \mu_t dt$ and $\nu = \nu_t dt$.

Proof. The idea here is similar to that of Brakke [15] (see also Ilmanen [100]). For completeness, we outline it here. Let $\phi \in C_0^2(M, \mathbb{R}_+)$. Then Lemma 8.2.1 implies

$$\begin{aligned}\frac{d}{dt}\int_M \phi e(u_i) &= -\int_M \phi\,|\partial_t u_i|^2 - \int_M \nabla\phi\cdot\langle\nabla u_i, \partial_t u_i\rangle \\ &\leq \int_M \phi(x)\,|\nabla u_i|^2 \leq C(\phi)E_0,\end{aligned}$$

where

$$C(\phi) = \sup_{\phi>0}\frac{|\nabla\phi|^2}{2\phi} \leq \sup_{\phi>0}|\nabla^2\phi| > 0.$$

Hence

$$\int_M \phi e(u_i(t)) - C(\phi)E_0 t$$

is monotonically decreasing with respect to $t > 0$. Let $B \subset \mathbb{R}_+$ be a countable dense subset. By the weak compactness of Radon measures with locally bounded mass, and a diagonal process, we can assume that

$$e(u_i)(x,t)\,dx \to \mu_t,\ \forall t \in B.$$

Now, let $\{\phi_i\}_{i\geq 1}$ be a countable dense subset in $C_0^2(M, \mathbb{R}_+)$. By the monotonicity of $\int_M \phi e(u_i(t)) - C(\phi)E_0 t$, there exists a co-countable set $C \subset R_+$ such that for any $t \in C$ and $i \geq 1$, $\mu_s(\phi_i)$ is continuous at t as a function of $s \in B$. For any fixed $t \in C$, there exists a further subsequence $i_j \to \infty$ and a limit μ_t such that $\mu_{i_j} \to \mu_t$. Hence $\{\mu_s(\phi_i)\}_{s\in B\cup\{t\}}$ is continuous at t, for all $i \geq 1$. Hence μ_t is uniquely determined by μ_s for $s \in B$. Therefore $\mu_i \to \mu_t$. Note that $\mathbb{R}_+ \setminus C$ is countable, we can do another diagonal process to show the result on $\mathbb{R}_+$. □

8.3 Stratification of the concentration set

In this section, we will present a stratification result by [136] for the energy concentration set Σ. Similar to the steady case discussed in Chapter 4, we consider the space, $T_z\mu$, of all tangent measures of μ for $z \in \Sigma$. We show that for any $\mu^0 \in T_z\mu$, $\mu^0 \lfloor \mathbb{R}^{n+1}_-$ is invariant under the parabolic dilation $\mathcal{P}_\lambda$ for all $\lambda > 0$. Hence we associate μ^0 a nonnegative integer d, which is the dimension of the largest $\mathcal{P}_\lambda$-invariant subspace which is a subset of

$$M\left(\Theta^n(\mu^0)\right) = \left\{z \in \mathbb{R}^{n+1} : \Theta^n\left(\mu^0, z\right) = \Theta^n(\mu^0, 0)\right\}.$$

Using this d, we can then stratify Σ accordingly. The proof of the stratification is based on an extension of Federer's dimension reduction argument (see Chapter 2 and Chapter 4). A similar scheme has been carried out by White [212] in an abstract setting with applications to mean curvature flows.

For simplicity, we assume $M = \mathbb{R}^n$. We introduce some additional notations. Define the parabolic dilation by

$$\mathcal{P}_{z_0,\lambda}(x,t) = \left(\frac{x - x_0}{\lambda}, \frac{t - t_0}{\lambda^2}\right) \text{ for } z_0 = (x_0, t_0) \in \mathbb{R}^{n+1} \text{ and } \lambda > 0.$$

Also define the Euclidean dilation by

$$\mathcal{D}_{x_0,\lambda}(x) = \frac{x - x_0}{\lambda}, \text{ for } x_0 \in \mathbb{R}^n \text{ and } \lambda > 0.$$

Denote

$$\mathbb{R}^{n+1}_+ = \mathbb{R}^n \times \mathbb{R}_+ \text{ and } \mathbb{R}^{n+1}_- = \mathbb{R}^n \times \mathbb{R}_-.$$

The Hausdorff dimension of $S \subset \mathbb{R}^{n+1}$ is the Hausdorff dimension with respect to the parabolic metric δ.

We write

$$\Sigma = \{(\Sigma_t, t) : 0 < t < \infty\} \text{ with } \Sigma_t = \Sigma \cap \{t\}.$$

For $\epsilon_i > 0$, let $u_i : \mathbb{R}^{n+1}_+ \to \mathbb{R}^L$ solve (8.8) with $\epsilon = \epsilon_i$. It follows from §7.6 and §8.2 that for $0 < R_2 \leq R_1 < \frac{\sqrt{t_0}}{2}$,

$$\begin{aligned} &\Phi(u_i, z_0, R_1) - \Phi(u_i, z_0, R_2) \\ \geq\ & \frac{1}{2}\int_{R_2}^{R_1}\int_{\mathbb{R}^n} \frac{|2(t_0-t)\partial_t u_i - (x-x_0)\nabla u_i|^2}{t_0 - t} G_{z_0}\, dz. \end{aligned} \tag{8.22}$$

By Lemma 8.2.9, we may assume that

$$e(u_i)(x,t)\, dxdt \to \mu \equiv \mu_t\, dt$$

as convergence of Radon measures in $\mathbb{R}^{n+1}_+$, for some nonnegative Radon measures $\{\mu_t\}_{t>0}$ on $\mathbb{R}^n$. By (8.2.1) and (8.2.2), we have

$$\sup_{(z,r)\in\mathbb{R}^{n+1}_+\times\mathbb{R}_+} r^{-n}\mu\left(P_r(z)\right) < \infty, \tag{8.23}$$

and

$$\Theta^n(\mu, z_0) = \lim_{R\downarrow 0} R^2 \int_{t=t_0-R^2} G_{z_0}\, d\mu_{t_0} \tag{8.24}$$

exists for all $z_0 = (x_0, t_0) \in \mathbb{R}^{n+1}_+$. For $z_0 \in \Sigma$ and $\lambda_i \downarrow 0$, define the parabolic rescaling of μ, $\mathcal{P}_{z_0,\lambda_i^{-1}}(\mu)$, by

$$\mathcal{P}_{z_0,\lambda_i}(\mu)(A) = \lambda_i^{-n}\mu\left(\mathcal{P}_{z_0,\lambda_i}(A)\right),\ \forall\ \text{Borel } A \subset \mathbb{R}^{n+1}.$$

Then by (8.23) there is a subsequence $\lambda_{i'}$ of λ_i and a nonnegative Radon measure μ^0 on $\mathbb{R}^{n+1}$ such that

$$\mathcal{P}_{z_0,\lambda_{i'}} \to \mu^0$$

as convergence of Radon measures on $\mathbb{R}^{n+1}$.

Definition 8.3.1 For any $z_0 \in \Sigma$, the tangent measure cone of μ at z_0, $T_{z_0}(\mu)$, consists of all nonnegative Radon measures on $\mathbb{R}^{n+1}$ that are given by

$$T_{z_0}(\mu) = \left\{\mu^0 :\ \text{there exists } r_i \downarrow 0 \text{ so that } \mathcal{P}_{z_0,\lambda_i}(\mu) \to \mu^0\right\}.$$

Since for any $z_0 \in \Sigma$ and $\mu^0 \in T_{z_0}(\mu)$ $\mu^0 = \mu^0_s\, ds$, we have for any $z = (x,t) \in \mathbb{R}^{n+1}$

$$\Theta^n(\mu^0, z, r) \equiv r^2 \int_{s=t-r^2} G_z(y,s)\, d\mu^0_s(y)$$

is monotonically nondecreasing with respect to r so that

$$\Theta^n(\mu^0, z) = \lim_{r\downarrow 0}\Theta^n(\mu^0, z, r)$$

exists and is upper semicontinuous for $z = (x,t) \in \mathbb{R}^{n+1}_+$.

Lemma 8.3.2 *For any $z_0 \in \Sigma$ and $\mu^0 \in T_{z_0}(\mu)$, $\mu^0\lfloor\mathbb{R}^{n+1}_-$ is invariant under all parabolic dilation, i.e.,*

$$\mathcal{P}_\lambda\left(\mu^0\lfloor\mathbb{R}^{n+1}_-\right) = \mu^0\lfloor\mathbb{R}^{n+1}_-. \tag{8.25}$$

Proof. Since $\mu^0 = \mu_t^0\,dt$, we have

$$\begin{aligned}
\mathcal{P}_\lambda\left(\mu^0\lfloor\mathbb{R}_-^{n+1}\right) &= \mathcal{P}_\lambda\left(\{(\mu_t^0, t) : t \le 0\}\right)\\
&= \left\{\left(\mathcal{D}_\lambda(\mu_t^0), \lambda^2 t\right) : t \le 0\right\}\\
&= \left\{\left(\mathcal{D}_\lambda(\mu^0_{\frac{t}{\lambda^2}}), t\right) : t \le 0\right\},
\end{aligned}$$

where

$$\mathcal{D}_\lambda(\mu_t^0)(A) = \lambda^{2-n}\mu_t^0(\mathcal{D}_\lambda A), \text{ for any borel } A \subset \mathbb{R}^n.$$

To prove (8.25), it suffices to show that for any $\lambda > 0$

$$\mathcal{D}_\lambda\left(\mu^0_{\frac{t}{\lambda^2}}\right) = \mu_t^0, \ \forall t \le 0. \tag{8.26}$$

Since $\lambda > 0$ is arbitrary, it suffices to prove (8.26) at $t = -1$, namely

$$\lambda^{n-2}\int_{\mathbb{R}^n} \phi(\lambda x)G(\lambda x, -1)\,d\mu^0_{-\lambda^{-2}}(x) = \int_{\mathbb{R}^n} \phi(x)G(x,-1)\,d\mu^0_{-1}(x), \tag{8.27}$$

for any $\phi \in C_0^1(\mathbb{R}^n)$ and $G = G_{(0,0)}$.

Note that there exists $\lambda_i \downarrow 0$ such that $v_i(x,t) = u_i(x_0 + \lambda_i x, t_0 + \lambda_i^2 t)$ satisfies (8.8), with ϵ_i replaced by $\overline{\epsilon}_i = \frac{\epsilon_i}{\lambda_i}$, and

$$e(v_i)(x,t)\,dx \equiv e_{\overline{\epsilon}_i}(v_i)(x,t)\,dx \to \mu_t^0, \ \forall t \in \mathbb{R}_-$$

as convergence of Radon measures on $\mathbb{R}^n$. Then for any $R > 0$,

$$\begin{aligned}
R^2\int_{t=-R^2} G\,d\mu_t^0 &= \lim_{i\to\infty} R^2\int_{t=-R^2} e(v_i)(x,t)G(x,t)\,dx\\
&= \lim_{\lambda_i\downarrow 0}\left\{(R\lambda_i)^2\int_{t=t_0-R^2\lambda_i^2} G_{z_0}(x,t)\,d\mu_t\right\}\\
&= \Theta^n(\mu, z_0).
\end{aligned}$$

This and (8.22) imply that for any $0 < r_1 < r_2 < \infty$,

$$\lim_{i\to\infty}\int_{t=-r_1^2}^{t=-r_2^2}\int_{\mathbb{R}^n} |x\cdot\nabla v_i + 2t\partial_t v_i|^2\,G = 0. \tag{8.28}$$

Note that (8.27) follows if we can show

$$\lim_{i\to\infty}\frac{d}{d\lambda}\left(\lambda^{n-2}\int_{t=-1}\phi(\lambda x)G(\lambda x,-1)e(v_i)(x,-\lambda^{-2})\right) = 0. \tag{8.29}$$

Direct calculations imply

$$\begin{aligned}
& \frac{d}{d\lambda}\left(\lambda^{n-2}\int_{t=-1}\phi(\lambda x)G(\lambda,x,-1)e(v_i)(x,-\lambda^{-2})\right) \\
= & \frac{d}{d\lambda}\left(\int_{t=-1}\phi(x)G(x,t)e(v_i^\lambda)(x,t)\,dx\right) \\
= & -\int_{t=-1}G\nabla\phi\nabla v_i^\lambda\cdot\frac{dv_i^\lambda}{d\lambda}-\int_{t=-1}\phi\left(\partial_t v_i^\lambda-\frac{1}{2}x\cdot\nabla v_i^\lambda\right)\cdot\frac{dv_i^\lambda}{d\lambda}G \\
= & -\frac{1}{2\lambda}\int_{t=-\lambda^2}\left|y\cdot\nabla v_i+2t\partial_t v_i\right|^2 G\phi\left(\frac{y}{\lambda}\right) \\
& -\int_{t=-\lambda^2}\nabla\phi\left(\frac{y}{\lambda}\right)G\nabla v_i\cdot(y\cdot\nabla v_i+2t\partial_t v_i),
\end{aligned}$$

where $v_i^\lambda(x,t)=v_n(\lambda^{-1}x,\lambda^{-2}t)$ and $e(v_i^\lambda)(x,t)=e_{\frac{\epsilon_i}{\lambda}}(v_i^\lambda)(x,t)$. Hence integrating this identity from 1 to λ and using (8.28) gives (8.27). □

By Lemma 8.3.2, we have that for any $z_0\in\Sigma$ and $\mu^0\in T_{z_0}\mu$,

$$R^2\int_{t=-R^2}G\,d\mu_t^0=\Theta^n(\mu^0,0)=\Theta^n(\mu,z_0),\ \forall R>0, \tag{8.30}$$

and for any $z\in\mathbb{R}^{n+1}_-$ and $\lambda>0$,

$$\Theta^n\left(\mu^0,z\right)=\Theta^n\left(\mu_0,\mathcal{P}_\lambda(z)\right). \tag{8.31}$$

In general, we have

Lemma 8.3.3 *For $z_0\in\Sigma$ and $\mu^0\in T_{z_0}(\mu)$, we have*
(1) $\Theta^n(\mu^0,z)\le\Theta^n(\mu^0,0)$, $\forall z\in\mathbb{R}^{n+1}$.
(2) If $z\in\mathbb{R}^{n+1}$ satisfies $\Theta^n(\mu^0,z)=\Theta^n(\mu^0,0)$, then

$$\Theta^n\left(\mu^0,z+v\right)=\Theta^n\left(\mu^0,z+\mathcal{P}_\lambda v\right),\ \forall\lambda>0,\ v\in\mathbb{R}^{n+1}_-. \tag{8.32}$$

Proof. (1) For $\mu^0\in T_{z_0}(\mu)$, there exists $r_i\downarrow 0$ such that $\mathcal{P}_{z_0,r_i}(\mu)\to\mu^0$. For any $r>0$ and $z=(x,t)\in\mathbb{R}^{n+1}$, we have

$$\begin{aligned}
\Theta^n(\mu^0,z) &\le \Theta^n(\mu^0,z,r) \\
&= \lim_{r_i\downarrow 0}\Theta^n\left(\mathcal{P}_{z_0,r_i}(\mu),z,r\right) \\
&= \lim_{r_i\downarrow 0}\Theta^n\left(\mu,z_0+(r_ix,r_i^2t),r_i^2r\right) \\
&\le \Theta^n(\mu,z_0)=\Theta^n(\mu^0,0),
\end{aligned}$$

where we have used the upper semicontinuity of $\Theta^n(\mu,\cdot,\cdot)$ with respect to the last two variables.

(2) From the proof of (1), we see that if $\Theta^n(\mu^0,z)=\Theta^n(\mu^0,0)$ then the inequalities are all equalities. In particular, $\Theta^n(\mu^0,z,r)$ is constant with respect to $r>0$. Applying the proof of lemma 8.3.2, we see that $\Theta^n(\mu^0,z+v)=\Theta^n\left(z+\mathcal{P}_\lambda(v)\right)$ for

any $v \in \mathbb{R}^{n+1}_-$ and $\lambda > 0$. □

For any $z_0 \in \Sigma$ and $\mu^0 \in T_{z_0}(\mu)$, denote

$$M\left(\Theta^n(\mu^0,\cdot)\right) \equiv \left\{z \in \mathbb{R}^{n+1} : \Theta^n(\mu^0, z) = \Theta^n(\mu^0, 0)\right\},$$

$$V\left(\Theta^n(\mu^0,\cdot)\right) \equiv M\left(\Theta^n(\mu^0,\cdot)\right) \cap \{t=0\},$$

and

$$\begin{aligned} W\left(\Theta^n(\mu^0,\cdot)\right) &= \{x \in \mathbb{R}^n : \Theta^n\left(\mu^0,(y,s)\right) = \Theta^n\left(\mu^0,(x+y,s)\right) \\ &\quad \text{for all } (y,s) \in \mathbb{R}^{n+1}_-\}. \end{aligned}$$

Then we have

Proposition 8.3.4 *For $z_0 \in \Sigma$ and $\mu^0 \in T_{z_0}(\mu)$, we have*

$$V\left(\Theta^n(\mu^0,\cdot)\right) = W\left(\Theta^n(\mu^0,\cdot)\right).$$

In particular, both $V\left(\Theta^n(\mu^0,\cdot)\right)$ and $W\left(\Theta^n(\mu^0,\cdot)\right)$ are linear subspaces of $\mathbb{R}^n$. Moreover, $M\left(\Theta^n(\mu^0,\cdot)\right)$ is either $V\left(\Theta^n(\mu^0,\cdot)\right)$ or $V\left(\Theta^n(\mu^0,\cdot)\right) \times (-\infty, a]$ for some $0 \le a \le \infty$ and $\Theta^n(\mu^0,\cdot)$ is time-independent up to $t = a$.

Proof. It is clear that $W\left(\Theta^n(\mu^0,\cdot)\right) \subset V\left(\Theta^n(\mu^0,\cdot)\right)$, $V\left(\Theta^n(\mu^0,\cdot)\right)$ is closed under scalar multiplications, and $mW\left(\Theta^n(\mu^0,\cdot)\right) \subset W\left(\Theta^n(\mu^0,\cdot)\right)$ for any positive integer m. For any $x \neq 0 \in V\left(\Theta^n(\mu^0,\cdot)\right)$, we have that for any $v \in \mathbb{R}^{n+1}_-$ and $\lambda > 0$,

$$\begin{aligned} \Theta^n\left(\mu^0,(x,0)+v\right) &= \Theta^n\left(\mu^0,(x,0)+\mathcal{P}_\lambda v\right) \\ &= \Theta^n\left(\mu^0,\mathcal{P}_{\lambda^{-1}}\left((x,0)+\mathcal{P}_\lambda v\right)\right) \\ &= \Theta^n\left(\mu^0,\mathcal{P}_{\lambda^{-1}}(x,0)+v\right) \end{aligned} \tag{8.33}$$

Since $v - \mathcal{P}_{\lambda^{-1}}(x,0) \in \mathbb{R}^{n+1}_-$, by replacing v by $v - \mathcal{P}_{\lambda^{-1}}(x,0)$ we have

$$\Theta^n\left(\mu^0,(x,0)+v-\mathcal{P}_{\lambda^{-1}}(x,0)\right) = \Theta^n\left(\mu^0,v\right). \tag{8.34}$$

Taking λ to 0 and using the upper semicontinuity of $\Theta^n(\mu^0,\cdot)$, (8.33) and (8.34) imply

$$\Theta^n\left(\mu^0,v\right) = \Theta^n\left(\mu^0,(x,0)+v\right), \ \forall v \in \mathbb{R}^{n+1}_-.$$

This gives $V\left(\Theta^n(\mu^0,\cdot)\right) \subset W\left(\Theta^n(\mu^0,\cdot)\right)$ so that $V\left(\Theta^n(\mu^0,\cdot)\right) = W\left(\Theta^n(\mu^0,\cdot)\right)$ and is a linear subspace of $\mathbb{R}^n$.

Suppose that $X = (x,t) \in M\left(\Theta^n(\mu^0,\cdot)\right)$ with $t < 0$. Then for any $Y = (y,s)$ with $s \le t$ and $\lambda > 0$, we have

$$\Theta^n\left(\mu^0,\mathcal{P}_{\lambda^{-1}}(Y)\right) = \Theta^n\left(\mu^0,Y\right) = \Theta^n\left(\mu^0,X+\mathcal{P}_{\lambda^{-1}}(Y-X)\right).$$

Note that $s\lambda^{-2} \le s \le t$ for $\lambda < 1$. Hence replacing Y by $\mathcal{P}_\lambda(Y)$ yields

$$\Theta^n\left(\mu^0,Y\right) = \Theta^n\left(\mu^0,X+Y-\mathcal{P}_{\lambda^{-1}}(X)\right).$$

Taking λ into zero, we have $\Theta^n\left(\mu^0, Y\right) \leq \Theta^n\left(\mu^0, X+Y\right)$. By substituting Y by $Y+\mathcal{P}_{\lambda^{-1}}(X)$, we also get

$$\Theta^n\left(\mu^0, Y+\mathcal{P}_{\lambda^{-1}}(X)\right)=\Theta^n\left(\mu^0, X+Y\right),$$

this implies $\Theta^n\left(\mu^0, Y\right) \geq \Theta^n\left(\mu^0, X+Y\right)$ by taking $\lambda \rightarrow \infty$. Therefore we have that for any $X=(x, t) \in M\left(\Theta^n(\mu^0, \cdot)\right)$ with $t<0$,

$$\Theta^n\left(\mu^0, Y\right)=\Theta^n\left(\mu^0, X+Y\right), \text{ for all } Y=(y, s),\ s \leq t. \tag{8.35}$$

By choosing $Y=(m-1) X$, this gives

$$\begin{aligned} \Theta^n\left(\mu^0, 0\right) & = \Theta^n\left(\mu^0,(m x, m t)\right)=\Theta^n\left(\mu^0, \mathcal{P}_m(m x, m t)\right) \\ & = \Theta^n\left(\mu^0,(x, \frac{t}{m})\right) \leq \Theta^n\left(\mu^0,(x, 0)\right). \end{aligned}$$

Therefore, $(x, 0) \in V\left(\Theta^n(\mu^0, \cdot)\right)=W\left(\Theta^n(\mu^0, \cdot)\right)$ and $(0, t) \in M\left(\Theta^n(\mu^0, \cdot)\right)$. In particular, $\Theta^n\left(\mu^0, \cdot\right)$ is time-independent up to $t=0$.

If $X=(x, t) \in M\left(\Theta^n(\mu^0, \cdot)\right)$ with $t>0$, then we can prove similarly that $\Theta^n\left(\mu^0, \cdot\right)$ is time-independent up to t. Let $t=a \geq 0$ be the maximal number such that $\Theta^n\left(\mu^0, \cdot\right)$ is time-independent up to $t=a$. Then we have $M\left(\Theta^n(\mu^0, \cdot)\right)= V\left(\Theta^n(\mu^0, \cdot)\right) \times(-\infty, a]$. □

Definition 8.3.5 For $z_0 \in \Sigma$ and $\mu^0 \in T_{z_0}(\Sigma)$, define $\dim\left(\Theta^n(\mu^0, \cdot)\right)$ by

$$=\begin{cases}\dim\left(V(\Theta^n(\mu^0, \cdot))\right)+2 & \text{if } M\left(\Theta^n(\mu^0, \cdot)\right)=V\left(\Theta^n(\mu^0, \cdot)\right) \times \mathbb{R} \\ \dim\left(V(\Theta^n(\mu^0, \cdot))\right) & \text{otherwise.}\end{cases}$$

Now we are ready to prove

Theorem 8.3.6 *For $0 \leq k \leq n$, let*

$$\Sigma_k=\left\{z_0 \in \Sigma: \dim\left(\Theta^n(\mu^0, \cdot)\right) \leq k,\ \forall \mu^0 \in T_{z_0}(\mu)\right\}.$$

Then $\dim_{\mathcal{P}}(\Sigma_k) \leq k$ for $0 \leq k \leq n$, and Σ_0 is discrete. In particular,

$$\Sigma=\Sigma_0 \cup\left(\Sigma_1 \backslash \Sigma_0\right) \cup \cdots \cup\left(\Sigma_n \backslash \Sigma_{n-1}\right),$$

and for $\mathcal{P}^n$ a.e. $z \in \Sigma$, there exists at least one $\mu^0 \in T_z(\mu)$ such that

$$\mu^0=\Theta^n(\mu, z)\left((H^{n-2}\lfloor T_{n-2}) \times(L^1\lfloor\mathbb{R})\right),$$

where $T_{n-2} \subset \mathbb{R}^n$ is a $(n-2)$-plane.

Proof of Theorem 8.3.6:
This is essentially an extension of Federer's dimension reduction argument. It suffices to show that if $\mathcal{P}^d(\Sigma_k)>0$ then $d \leq k$. Thus we only consider such a d. First, denote $A_{z, r}=\mathcal{P}_{z, r}(A)$ for $A \subset \mathbb{R}^{n+1}$, $z \in \mathbb{R}^{n+1}$ and $\lambda>0$. Let

$$\begin{aligned} \mathcal{C} = & \{V \times \mathbb{R}, \text{ or } V: V \subset \mathbb{R}^n \text{ is a linear subspace of } \dim \leq k-2\} \\ & \cup\{V \times \mathbb{R}_-: V \subset \mathbb{R}^n \text{ is a linear subspace of } \dim \leq k\}. \end{aligned}$$

Then we have

Claim. For any $z_0 \in \Sigma_k$ and $r > 0$ there exists $\eta = \eta(z, \epsilon) > 0$ such that for any $\rho \in (0, \epsilon)$

$$\left(\{w \in P_\rho(z) : \Theta^n(\mu, w) \geq \Theta^n(\mu, z) - \eta\}\right)_{z,\rho} \subset \epsilon\text{-neighborhood of } C \tag{8.36}$$

for some $C \in \mathcal{C}$, where ϵ-neighborhood is taken with respect to the parabolic metric δ.

For, otherwise, there exist $\epsilon_0 > 0$, $z_0 \in \Sigma_k$, and $\rho_i, \eta_i \downarrow 0$ such that

$$\begin{aligned} B_i &\equiv \{z \in P_1(0) : \Theta^n\left(\mathcal{P}_{z_0,\rho_i}(\mu), z\right) \geq \Theta^n\left(\mu, z_0\right) - \eta_i\} \\ &\not\subset \epsilon_0\text{-neighborhood of any } C \in \mathcal{C}. \end{aligned}$$

On the other hand, we may assume that $\mathcal{P}_{z_0,\rho_i}(\mu) \to \mu^0$ for some $\mu^0 \in T_{z_0}(\mu)$ and $B_i \to B$ in the Hausdorff distance for some $B \subset M\left(\Theta^n(\mu^0, \cdot)\right)$. By Proposition 8.3.4, we have that among all four possibilities of $M\left(\Theta^n(\mu^0, \cdot)\right)$, only $M\left(\Theta^n(\mu^0, \cdot\right) = V\left(\Theta^n(\mu^0, \cdot)\right) \times (-\infty, a]$ for some $a > 0$ does not belong to $\mathcal{C}$. However, for such a case we can still find $r_i \downarrow 0$ such that $\mathcal{P}_{r_i}(\mu^0) \to \mu^1$, and by the upper semicontinuity

$$\Theta^n(\mu^1, w) = \Theta^n(\mu^1, 0) = \Theta^n(\mu^0, 0), \ \forall w \in V\left(\Theta^n(\mu^0, \cdot)\right) \times \mathbb{R}$$

this implies that $M\left(\Theta^n(\mu^0, \cdot)\right) \subset M\left(\Theta^n(\mu^1, \cdot)\right) \in \mathcal{C}$. Therefore we get the desired contradiction and the claim holds.

We now proceed as follows. Decompose $\Sigma_k = \cup_{j,q \geq 1} \Sigma_{k,j,q}$, where $\Sigma_{k,j,q}$ denotes points in Σ_k such that $\Theta^n(\mu, \cdot) \in \left(\frac{q-1}{j}, \frac{q}{j}\right)$ and the conclusion of the above claim holds with $\eta = j^{-1}$ and $\epsilon = j^{-1}$. Therefore, for each j, there exists $q \geq 1$ such that $\mathcal{P}^d(\Sigma_{k,j,q}) > 0$. By the lower bound for the upper density ([55]), we know that there exist $z_j \in \Sigma_{k,j,q}$ and $r_i \downarrow 0$ such that

$$\mathcal{P}^{d,\infty}((\Sigma_{k,j,q})_{z_j,r_i}) \geq 10^{-d}, \tag{8.37}$$

where $\mathcal{P}^{d,\infty}$ denotes the outer d-dimensional Hausdorff measure with size ∞. Moreover, for each $z \in (\Sigma_{k,j,q})_{z_j,r_i}$ there exist $\epsilon_i \downarrow 0$ and $C_z \in \mathcal{C}$ such that

$$(\Sigma_{k,j,q})_{z_j,r_i} - z \subset \ \epsilon_i\text{-neighborhood of } C_z \text{ for some } C_z \in \mathcal{C}.$$

We may assume that $(\Sigma_{k,j,q})_{z_j,r_i} \to \Sigma_k^\infty$ as $i \to \infty$. Then we have

$$\Sigma_k^\infty - z \subset C_z \ \text{ for all } z \in \Sigma_k^\infty, \text{ and } \mathcal{P}^{d,\infty}\left(\Sigma_k^\infty\right) \geq 10^{-d}. \tag{8.38}$$

For any $C \in \mathcal{C}$, let $\Sigma_{k,l,C}^\infty = \left\{z \in \Sigma_k^\infty : \delta(C_z, C) \leq l^{-1}\right\}$. Then for each l there exists $C_l \in \mathcal{C}$ such that $\Sigma_{k,l}^\infty \equiv \Sigma_{k,l,C_l}^\infty$ has positive $\mathcal{P}^d$-measure. Therefore, there exist $z_l \in \Sigma_{k,l}^\infty$ and $\rho_l \downarrow 0$ such that

$$\mathcal{P}^{d,\infty}((\Sigma_{k,l}^\infty)_{z_l,\rho_l}) \geq 10^{-d}. \tag{8.39}$$

Assume that $C_l \to C_\infty \in \mathcal{C}$ and $(\Sigma_{k,l}^\infty)_{z_l,\rho_l} \to \Sigma^\infty$ as $l \to \infty$. Then $\Sigma^\infty \subset C_\infty$, $\mathcal{P}^d(\Sigma^\infty) > 0$, and $\Sigma^\infty - z \subset C_\infty$ whenever $z \in \Sigma^\infty$. In particular, we have $\Sigma^\infty \subset C_\infty \cap (-C_\infty)$. But we note that if $C_\infty = V_\infty \times \mathbb{R}_-$ for some vector subspace $V_\infty \subset \mathbb{R}^n$ then $C_\infty \cap (-C_\infty) = V_\infty$ so that $\mathcal{P}^d(V_\infty) > 0$ is equivalent to $H^d(V_\infty) > 0$ and hence $d \leq k$ (since V_∞ is at most k dimensional). For $C_\infty = V_\infty$ or $V_\infty \times \mathbb{R}$, we see $C_\infty \cap (-C_\infty) = C_\infty$ so that $\mathcal{P}^d(\Sigma^\infty) > 0$ implies $\mathcal{P}^d(C_\infty) > 0$ and hence $d \leq k$ again. This completes the proof. □

8.4 Blow up analysis in dimension two

In this section, we will present a proof of Theorem 8.1.1 and Theorem 8.1.2 for $n = 2$, which is a slight simplification of the original one given in [134] and makes use of the stratification result from §8.3. Besides its own interests, the proof for $n = 2$ also indicates some crucial ideas of the proof to be given in §8.5 for $n \geq 3$.

The goal is to show that if $u_i \not\to u$ strongly in $H^1(M \times \mathbb{R}_+, N)$ then we can extract a harmonic S^2 in the concentration set associated with u_i. Since the situation of Ginzburg-Landau heat flows is in fact slightly more complicate than the heat flow of harmonic maps, we decide to work on the former case.

Since the theorem and its proof are local, we assume for simplicity that $M = B_1 \subset \mathbb{R}^2$ and $T = 2$.

For $\epsilon_i \downarrow 0$, let $u_i = u_{\epsilon_i}$ be solutions of (8.8). Denote $e(u_i) = e_{\epsilon_i}(u_{\epsilon_i})$. Assume that there are two nonnegative Radon measures ν, η in $B_1 \times [0, 2]$ such that

$$e(u_i)\, dxdt \to \mu \equiv \frac{1}{2}|\nabla u|^2\, dxdt + \nu, \quad |\partial_t u_i|^2\, dxdt \to |\partial_t u|^2 + \eta$$

as convergence of Radon measures on $B_1 \times [0, 2]$.

Suppose that u_i does not converge to u in $H^1(B_1 \times [0, 2], \mathbb{R}^L)$, then by Lemma 8.2.7 we have that $\mathcal{P}^2(\Sigma) > 0$ and $\nu(B_1 \times [0, 2]) > 0$. Note also that

$$F = \left\{ z \in B_1 \times (0, 2] \mid \eta(\{z\}) \equiv \lim_{r\downarrow 0} \eta(P_r(z)) > 0 \right\}$$

is at most a countable subset. Therefore we can find $z_0 = (x_0, t_0) \in \Sigma$ such that

$$\Theta^{*,2}(\Sigma, z_0) = \overline{\lim}_{r\downarrow 0}\, r^{-2}\mathcal{P}^2(\Sigma \cap P_r(z_0)) \geq \frac{1}{4}, \tag{8.40}$$

$$\lim_{r\downarrow 0} r^{-2} \int_{P_r(z_0)} |\nabla u|^2 = 0, \tag{8.41}$$

and

$$\lim_{r\downarrow 0} \lim_{i\to\infty} \int_{P_r(z_0)} |\partial_t u_i|^2 = 0. \tag{8.42}$$

For $r_i \downarrow 0$, define $v_i(x, t) = u_i(z_0 + (r_i x, r_i^2 t)) : B_2 \times [-2, 0] \to \mathbb{R}^L$. Then v_i solves (8.8) in $B_2 \times [-2, 0]$, with ϵ_i replaced by $\overline{\epsilon}_i (= r_i^{-1}\epsilon_i)$, $v_i \to$ const weakly but not strongly in $H^1(B_2 \times [-2, 0], \mathbb{R}^L)$. It follows from the discussion of §8.3 that there is a nonnegative Radon measure ν_* in $B_2 \times [-2, 0]$ such that

$$e(v_i)\, dxdt \equiv e_{\overline{\epsilon}_i}(v_i)\, dxdt \to \nu_*$$

as convergence of Radon measures on $B_2 \times [-2, 0]$. Therefore we have $\nu_*(B_2 \times [-2, 0]) > 0$. Moreover, by the upper semicontinuity if we denote $\Sigma_* = \mathrm{supp}(\nu_*)$ then we have $(0, 0) \in \Sigma_*$ and

$$\mathcal{P}^2(\Sigma_*) \geq \lim_{i\to\infty} r_i^{-2}\mathcal{P}^2\left(\Sigma \cap P_{r_i}(z_0)\right) \geq \frac{1}{4}.$$

It also follows from §8.3 that we have that $\mathcal{P}_\lambda(\nu_*) = \nu_*$ for all $\lambda > 0$. Hence if we write $\nu_* = \nu_*^t\,dt$ on $B_2 \times [-2, 0]$ for a family of nonnegative Radon measures $\{\nu_*^t\}, -2 \le t \le 0$, on B_2, then

$$\nu_*^t = \sqrt{-t}\nu_*^{-1} \text{ for all } -2 \le t \le 0.$$

This also implies that for all $-2 \le t \le 0$,

$$\Sigma_*^t \equiv \Sigma_* \cap \{t\} = \sqrt{-t}\Sigma_*^{-1}$$

so that $\mathcal{P}^2(\Sigma_*) > 0$ implies that $0 < H^0(\Sigma_*^{-1}) < +\infty$, i.e. $\emptyset \neq \Sigma_*^{-1}$ is a finite set.

Now we claim that $\Sigma_*^{-1} = \{0\}$ and $\nu_*^{-1} = m\delta_0$ for some positive constant m.

In fact, Lemma 8.2.1 implies that for $\phi \in C_0^\infty(B_2)$ and $-1 \le t_1 < t_2 \le 0$,

$$\begin{aligned}&\int_{B_2} e(v_i(t_2))\phi - \int_{B_2} e(v_i(t_1))\phi \\ &= -\int_{t_1}^{t_2}\int_{B_2} |\partial_t v_i|^2\phi - \int_{t_1}^{t_2}\int_{B_2} \langle \nabla v_i, \partial_t v_i\rangle \cdot \nabla\phi.\end{aligned} \tag{8.43}$$

Note that by (8.42) we have

$$\int_{-1}^{0}\int_{B_2} |\partial_t v_i|^2 = \int_{t_0 - r_i^2}^{t_0}\int_{B_{r_i}(x_0)} |\partial_t u_i|^2 \to 0.$$

Hence the right hand side of (8.43) tends to zero as $i \to \infty$. Taking i to ∞, (8.43) yields

$$\nu_*^{t_1}(\phi) = \nu_*^{t_2}(\phi)$$

so that ν_*^t is independent of t for $-1 \le t \le 0$. On the other hand, we already prove that $\nu_*^t = \sqrt{-t}\nu_*^{-1}$ for $-1 \le t \le 0$. Thus we must have $\nu_*^t = m\delta_0$ for some positive constant m.

Hence for $r_0 > 0$ small there exist $\delta_i \downarrow 0$ and $x_i(\subset B_{r_0}) \to 0$ such that for $t_0 = -\frac{1}{2}$ we have

$$\int_{B_{\delta_i}(x_i)} e(v_i(t_0)) = \frac{\epsilon_0^2}{C(2)} = \max\left\{\int_{B_{\delta_i}(x)} e(v_i(t_0)) \mid x \in B_{r_0}\right\} \tag{8.44}$$

for some sufficiently large $C(2) > 0$ to be chosen later.

Define $w_i(x,t) = v_i(x_i + \delta_i x, -\frac{1}{2} + \delta_i^2 t)$, $x \in \Omega_i \equiv \delta_i^{-1}(B_{r_0} \setminus \{x_i\}), t \in [-4, 4]$. Then we have that w_i solves (8.8) on $\Omega_1 \times [-4, 4]$, with ϵ_i replaced by $\tilde{\epsilon}_i = \frac{\epsilon_i}{\delta_i r_i}$, and

$$\int_{B_1} e(w_i(0)) = \frac{\epsilon_0^2}{C(2)} = \max\left\{\int_{B_1(x)} e(w_i(0)) \mid x \in \Omega_i\right\}. \tag{8.45}$$

Since

$$\int_{-4}^{4}\int_{\Omega_i} |\partial_t w_i|^2 = \int_{t_0 - 4\delta_i^2}^{t_0 + 4\delta_i^2} |\partial_t v_i|^2 \le \int_{-1}^{0}\int_{B_1} |\partial_t v_i|^2 \to 0, \tag{8.46}$$

we can apply (8.43), with v_i replaced by w_i, to conclude that for i sufficiently large we have that for any $y \in \Omega_i$ and $t \in [-2, 2]$

$$\int_{B_2(y)} e(w_i(t)) \le \frac{16\epsilon_0^2}{C(2)} \le \epsilon_0^2, \tag{8.47}$$

provided that $C(2)$ is chosen to be sufficiently large. Hence by Lemma 8.2.3 we have

$$e(w_i)(z) \le C(\epsilon_0) \text{ for all } z \in \Omega_i \times [-2, 2]. \tag{8.48}$$

Since $\Omega_i \to \mathbb{R}^2$, by (8.48) we may assume $w_i \to w_\infty$ in $C^2_{\text{loc}}(\mathbb{R}^2 \times [-2, 2], \mathbb{R}^L)$. By (8.46) we have that $\partial_t w_\infty \equiv 0$ and hence $w_\infty(x, t) = w_\infty(x)$ for $t \in [-2, 2]$. By the conformal invariance of E and (8.45), we have

$$\frac{\epsilon_0^2}{C(2)} \le \int_{\mathbb{R}^2} e(w_\infty) < +\infty.$$

For the equation of w_∞, we have three possibilities:
(i) $\frac{\epsilon_i}{\delta_i r_i} \to 0$: it is well-known that w_∞ is a non constant harmonic map from $\mathbb{R}^2$ to N with finite energy that can be lifted to be a harmonic S^2 in N.
(ii) $\frac{\epsilon_i}{\delta_i r_i} \to \infty$: it is easy to see that w_∞ is a non constant harmonic function on $\mathbb{R}^2$ with finite energy. This is impossible.
(iii) $\frac{\epsilon_i}{\delta_i r_i} \to c_0 > 0$: it is easy to show that w_∞ is a non constant solution to

$$\Delta w_\infty + \frac{1}{c_0^2} f(w_\infty) = 0 \text{ in } \mathbb{R}^2 \tag{8.49}$$

with $\int_{\mathbb{R}^2} e_{c_0}(w_\infty) < +\infty$. This is impossible by the following simple lemma.

Lemma 8.4.1 *If $v \in C^\infty(\mathbb{R}^2, \mathbb{R}^L)$ satisfies*

$$\Delta v + \frac{1}{c^2} f(v) = 0 \text{ in } \mathbb{R}^2$$

for some $0 < c < +\infty$, and $\int_{\mathbb{R}^2} e_c(v) < +\infty$, then v is constant.

Proof. Let $\phi \in C_0^\infty(B_2)$ be such that $\phi \equiv 1$ in B_1, and define $\phi_i(x) = \phi(\frac{x}{i})$ for $i \ge 1$. Multiplying the equation of v by $\phi_i(x) x \cdot \nabla v$ and integrating over $\mathbb{R}^2$, it is not hard to get

$$\int_{\mathbb{R}^2} F(v) \le C \int_{B_{2i} \setminus B_i} e_c(v) \to 0$$

as $i \to \infty$. This implies $F(v) \equiv 0$ and hence $\Delta v \equiv 0$ on $\mathbb{R}^2$. On the other hand, we have $\int_{\mathbb{R}^2} |\nabla v|^2 < +\infty$. Therefore v is constant. □

8.5 Blow up analysis in dimensions $n \geq 3$

In this section, we will prove Theorem 8.1.1 and Theorem 8.1.2 for $n \geq 3$. Namely, if the strong convergence of u_i to u fails then there exists at least one harmonic S^2 in N. The presentation here is a slight improvement of the original one given by Lin-Wang [134], with a few more details provided.

For simplicity, we assume $M = B_2^n$ and $T = 2$. As in §8.5, we work on the Ginzburg-Landau heat flow (8.8).

We assume that there are two nonnegative Radon measures ν, η on $B_2^n \times [0,2]$ such that

$$e(u_i)\,dxdt \to \mu \equiv \frac{1}{2}|\nabla u|^2\,dxdt + \nu,\ |\partial_t u_i|^2\,dxdt \to |\partial_t u|^2\,dxdt + \eta$$

as convergence of Radon measures on $B_2^n \times [0,2]$.

Suppose that $u_i \not\to u$ strongly in $H^1(B_2^n \times [0,2], \mathbb{R}^L)$. Then by Lemma 8.2.7 we have that

$$\mathcal{P}^n(\Sigma) > 0 \text{ and } \nu(B_2^n \times [0,2]) > 0.$$

By Lemma 8.2.5 and Lemma 8.2.6, we can conclude that there is a subset $\Sigma_1 \subset \Sigma$, with $\mathcal{P}^n(\Sigma_1) = \mathcal{P}^n(\Sigma) > 0$ such that
(i) $\limsup_{r\downarrow 0} r^{-n}\mathcal{P}^n(\Sigma \cap P_r(z)) \geq 2^{-n}$ for all $z \in \Sigma_1$.
(ii) $\Theta^n(u,z) = \lim_{r\downarrow 0} r^{-n}\int_{P_r(z)} |\nabla u|^2 = 0$ for all $z \in \Sigma_1$.
(iii) $\epsilon_0^2 \leq \Theta^n(\nu,z) = \Theta^n(\mu,z) < +\infty$ for all $z \in \Sigma_1$.
(iv) $\Theta^n(\mu,z) = \Theta^n(\nu,z)$ is $\mathcal{P}^n$-approximate continuous on Σ_1, i.e. for any $\epsilon > 0$

$$\lim_{r\downarrow 0}\left(r^{-n}\mathcal{P}^n\left(\{w \in \Sigma_1 \cap P_r(z) : |\Theta^n(\mu,w) - \Theta^n(\mu,z)| > \epsilon\}\right)\right) = 0 \tag{8.50}$$

for all $z \in \Sigma_1$.

Moreover, we claim that
(v)

$$\lim_{r\downarrow 0}\lim_{i\to\infty} r^{2-n}\int_{P_r(z)} |\partial_t u_i|^2 = 0 \ \text{ for all } z \in \Sigma_1. \tag{8.51}$$

Note that if we denote $\tilde{\eta} \equiv |\partial_t u|^2\,dxdt + \eta$ then (8.51) is equivalent to

$$\Theta^{n-2}(\tilde{\eta}) \equiv \lim_{r\downarrow 0} r^{2-n}\tilde{\eta}(P_r(z)) = 0 \ \text{ for all } z \in \Sigma. \tag{8.52}$$

To prove (8.51), it suffices to show that for any $\epsilon > 0$,

$$F_\epsilon = \left\{z \in \Sigma \mid \lim_{r\downarrow 0}\lim_{i\to\infty} r^{2-n}\int_{P_r(z)} |\partial_t u_i|^2 \geq \epsilon\right\}$$

has $\mathcal{P}^{n-2}(F_\epsilon) < +\infty$. This can be done by employing Vitali's covering lemma as follows. For any $\delta > 0$, there are $\{z_i\} \subset \Sigma$ and $0 < r_i < \delta$ such that $\{P_{r_i}(z_i)\}$ are mutually disjoint, $\{P_{5r_i}(z_i)\}$ covers Σ, and there is a subsequence u_j of u_i so that

$$r_i^{2-n}\int_{P_{r_i}(z_i)} |\partial_t u_j|^2 \leq \epsilon,\ \forall j.$$

Hence

$$\begin{aligned}
\mathcal{P}_{5\delta}^{n-2}(F_\epsilon) &\le \sum_{i=1}^{\infty}(5r_i)^{n-2} = 5^{n-2}\sum_{i=1}^{\infty} r_i^{n-2} \\
&\le \frac{5^{n-2}}{\epsilon}\sum_{i=1}^{\infty}\int_{P_{r_i}(z_i)} |\partial_t u_j|^2 \\
&\le \frac{5^{n-2}}{\epsilon}\int_{\cup_{i=1}^{\infty} P_{r_i}(z_i)} |\partial_t u_j|^2 \\
&\le C(n,\epsilon)\int_0^2\int_{B_2} |\partial_t u_j|^2 \le C(n,\epsilon,E_0)
\end{aligned}$$

by the energy inequality. Taking δ to zero, this yields (8.51).

Pick a point $z_0 = (x_0,t_0) \in \Sigma_1$. For $r_i \downarrow 0$, let $v_i(x,t) = u_i(x_0 + r_i x, t_0 + r_i^2 t) : P_2 \to \mathbb{R}^L$, and $\mu_i = \mathcal{P}_{z_0,r_i}(\mu)$. Then we have that v_i solves (8.8) on P_2 with ϵ_i replaced by $\overline{\epsilon}_i = \frac{\epsilon_i}{r_i}$, and by a diagonal process we can assume that

$$v_i \to \text{constant weakly, but not strongly in } H^1(P_2,\mathbb{R}^L),$$

and

$$e(v_i)\,dxdt = e_{\overline{\epsilon}_i}(v_i)\,dxdt \to \mu_* \equiv \mu_*^t\,dt$$

as convergence of Radon measures on P_2 for a Radon measure $\mu_* = \mu_*^t\,dt$.

Denote $\Sigma_* = \text{supp}(\mu_*)$ and write $\Sigma_* = \cup_{t\in(-4,0]}\Sigma_*^t$. Then it follows from §8.2 and §8.3 that

$$\begin{aligned}
\Sigma_*^t &= \bigcap_{R>0}\left\{x\in B_2 \mid \lim_{i\to\infty}\int_{T_R((x,t))}\eta^2 e(v_i)G_{(x,t)} \ge \epsilon_0^2\right\} \\
&= \left\{x\in B_2 \mid \Theta^n(\mu_*,z) \ge \epsilon_0^2\right\}.
\end{aligned}$$

Moreover, (i) implies that

$$(0,0)\in\Sigma_*,\ \mathcal{P}^n(\Sigma_*) > 0,\ \mu_*(P_2) \ge \epsilon_0^2. \tag{8.53}$$

To proceed the proof, we need

Claim 1. For $t \in (-4,0]$, $H^{n-2}(supp(\mu_*^t)) > 0$.

Note that since $\Sigma_*^t = \text{supp}(\mu_*^t)$ for $-4 < t \le 0$, the claim is equivalent to $H^{n-2}(\Sigma_*^t) > 0$. We can prove this claim by contradiction. Suppose it were false. Then there is $t_0 \in (-4,0]$ such that $H^{n-2}(\Sigma_*^{t_0}) = 0$. Then for any $\epsilon > 0$ there exist $\delta > 0$ and a covering $\{B_{r_i}(x_i)\}_{i=1}^{\infty}, 0 < r_i \le \delta$, of $\Sigma_*^{t_0}$, with $x_i \in \Sigma_*^{t_0}$, such that

$$\sum_{i=1}^{\infty} r_i^{n-2} < \epsilon.$$

Since

$$\mu_*^{t_0}(B_1 \setminus \cup_{i=1}^{\infty} B_{r_i}(x_i))) = 0,$$

by a diagonal process we may assume that there is a subsequence u_j of u_i so that

$$\lim_{j\to\infty}\int_{B_1\setminus\cup_{i=1}^\infty B_{r_i}(x_i)} e(v_j(t_0)) = 0.$$

Moreover, by (8.15) we have

$$\begin{aligned} r_i^{2-n}\int_{B_{r_i}(x_i)} e(v_j(t_0)) &\leq e^{-1}r_i^2\int_{\{t=(t_0+r_i^2)-r_i^2\}}\eta^2 e(v_j(t))G_{(x_i,t_0+r_i^2)}(\cdot,t)\\ &= e^{-1}\Phi(v_j,(x_i,t_0+r_i^2),r_i)\\ &\leq e^{-1}\Phi(v_j,(x_i,t_0+r_i^2),R)+CK(R-r_i)\\ &\leq e^{-1}R^{2-n}\int_{\{t=t_0+r_i^2-R^2\}} e(v_j(t))+C(K)(R-r_i)\\ &\leq C(R,K) \end{aligned}$$

for some $R >> r_i$. Hence

$$\begin{aligned} \int_{\cup_{i=1}^\infty B_{r_i}(x_i)} e(v_j(t_0)) &\leq \sum_{i=1}^\infty\int_{B_{r_i}(x_i)} e(v_j(t_0))\\ &\leq C\sum_{i=1}^\infty r_i^{n-2}\leq C\epsilon \end{aligned}$$

so that if we choose $\epsilon < \frac{\epsilon_0^2}{16C}$ then for sufficiently large j we have

$$\int_{B_1} e(v_j(t_0)) \leq \frac{\epsilon_0^2}{8}.$$

On the other hand, (8.51) implies that

$$\lim_{i\to\infty}\int_{P_2}|\partial_t v_i|^2 = 0. \tag{8.54}$$

As in §8.4, we have that for any $\phi\in C_0^\infty(B_2)$ and $-4 < t_1 < t_2 \leq 0$,

$$\begin{aligned} &\int_{B_2} e(v_i(t_2))\phi - \int_{B_2} e(v_i(t_1))\phi\\ = &-\int_{t_1}^{t_2}\int_{B_2}|\partial_t v_i|^2\phi - \int_{t_1}^{t_2}\int_{B_2}\langle\nabla v_i,\partial_t v_i\rangle\cdot\nabla\phi. \end{aligned}$$

Therefore, taking i to ∞, we obtain that $\nu_*^t(\phi)$ is independent of $t\in(-4,0]$. Thus we obtain

$$\int_{P_1} e(v_i)\leq \sup_{t\in[-1,0]}\int_{B_1} e(v_i(t))\leq 4\int_{B_1} e(v_j(t_0))\leq\frac{\epsilon_0^2}{2}.$$

This contradicts the fact that $(0,0)\in\Sigma_*$.

In fact, by employing Lemma 8.3.2 we have that $\mathcal{P}_\lambda(\mu_*)=\mu_*$ for any $\lambda>0$. In particular, $\mu_*^t=\sqrt{-t}\mu_*{-1}$ for $-4\leq t\leq 0$. This, combined with the fact that μ_*^t is independent of t, yields that ν_t^{-1} is an Euclidean cone measure and hence Σ_*^{-1} is an Euclidean cone.

Note that, since $\Theta^n(\nu, z) = \Theta^n(\mu, z)$ is $\mathcal{P}^n$-approximate continuous at z_0 for $z \in \Sigma$ and $\Theta^n(\mu, z)$ is upper semicontinuous with respect to (μ, z), we conclude that for $\mathcal{P}^n$ a.e. $z \in \Sigma_*$,

$$\Theta^n(\mu_*, z) \geq \Theta^n(\mu, z_0).$$

On the other hand, it follows from the discussion in §8.3 that

$$\Theta^n(\mu_*, z) \leq \Theta^n(\mu_*, 0) = \Theta^n(\mu, z_0).$$

Hence

$$\Theta^n(\mu_*, z) = \Theta^n(\mu_*, 0) \quad \text{for } \mathcal{P}^n \text{ a.e. } z \in \Sigma_*. \tag{8.55}$$

Therefore Proposition 8.3.4 implies that there exist a set $\Sigma_2 \subset \mathbb{R}^n$ with $H^{n-2}(\Sigma_2) = 0$ and $(n-2)$-dimensional plane $P \subset \mathbb{R}^n$ such that $\Sigma_2 \cap P = \emptyset$ and

$$\Sigma_*^t = \Sigma_2 \cup P \text{ for any } t \in (-1, 0]. \tag{8.56}$$

In fact, $\Sigma_2 = \emptyset$. For, otherwise, pick $z \in \Sigma_2 \setminus P$. Then Proposition 8.3.4 implies that

$$\Theta^n(\mu, z+y) = \Theta^n(\mu, z) \text{ for all } y \in P,$$

so that $\{z\} \cup P \subset \Sigma_*^{-1}$ and hence $H^{n-2}(\Sigma_*^{-1}) = +\infty$ which contradicts Theorem 7.2.4.

We would like to point out that (8.56) can be derived from the following geometric lemma, which is similar to Lemma 4.2.6 for harmonic maps in §4. We will postpone its proof to §9.1 when we will prove the time slice rectifiability of Σ.

Denote

$$T_1 = \left\{t \in [-4, 0] \mid \Sigma_1^t = \Sigma_1 \cap \{t\} \neq \emptyset\right\},$$

and

$$T_2 = \left\{t \in [-4, 0] \mid \lim_{i\to\infty} \int_{B_2} |\partial_t u_i|^2 < +\infty\right\}.$$

Note that by Fatou's lemma, we have that $L^1(T_2) = L^1([-4, 0]) = 4$ has full Lebesgue measure of $[-4, 0]$.

Lemma 8.5.1 *For L^1 a.e. $t \in T_1 \cap T_2$, there exist subset $E_t \subset \Sigma_t$, with $H^{n-2}(E_t) = H^{n-2}(\Sigma_t)$, and a number $s \in (0, \frac{1}{2})$ depending only on n such that for any $x \in E_t$, there is $r_x > 0$ so that for any $0 < r < r_x$, there are $(n-2)$-points $\{x_1, \cdots, x_{n-2}\} \subset \Sigma_1 \cap B_r(x)$ and $\epsilon(r) > 0$ with $\lim_{r\downarrow 0} \epsilon(r) = 0$ such that*
(i)

$$\Theta^{n-2}(\mu_t, x_i)(\equiv \lim_{r\downarrow 0} r^{2-n} \mu_t(B_r(x_i))) \geq \Theta^{n-2}(\mu_t, x) - \epsilon(r) \text{ for all } i = 1, \cdots, n-2.$$

(ii) $|x_1| \geq sr$ *and for any* $j \in \{2, \cdots, n-2\}$,

$$dist(x_j, x + V_{k-1}) \geq sr, \quad \text{where } V_{k-1} = span\{x_1 - x, \cdots, x_{k-1} - x\}.$$

We identify $\mathbb{R}^{n-2}$ as $\{(0,0)\} \times \mathbb{R}^{n-2}$. By a change of coordinate systems, we may assume that $P = \mathbb{R}^{n-2} \subset \mathbb{R}^n$ and write $X = (x, y) \in \mathbb{R}^n$ for $x \in \mathbb{R}^2$ and $y \in \mathbb{R}^{n-2}$.

Summarizing the properties of μ_* gives

$$\mu_*(X,t) = \Theta^n(\mu, z_0) \left(H^{n-2} \lfloor \mathbb{R}^{n-2}\right)(y) \times \left(\mathcal{P}^2 \lfloor \mathbb{R}\right)(t) \tag{8.57}$$

for all $X = (x, y) \in B_2$ and $t \in [-4, 0]$.
Let's pick $(n-2)$ linearly independent points $\{\xi_1, \cdots, \xi_{n-2}\} \subset \mathbb{R}^{n-2}$ that span $\mathbb{R}^{n-2}$. Note that we have

$$\Theta^n(\mu_*, (\xi_j, 0), R) = \Theta^n(\mu_*, (\xi_j, 0)) = \Theta^n(\mu, z_0), \ \forall R > 0 \text{ for } j = 1, \cdots, n-2, \tag{8.58}$$

where

$$\Theta^n\left(\mu_*, (\xi_j, 0), R\right) = \int_{T_R} G_{(\xi_j,0)}\, d\mu_*.$$

Since $e(v_i) \to \mu_*$ in P_2, using (8.58) and applying the monotonicity inequality (8.14) of v_i with centers $(x_j, 0)$ for $j = 1, \cdots, n-2$, we can easily see that for any $\rho > 0$

$$\lim_{j\to\infty} \int_\rho^1 \frac{dr}{r} \int_{T_r} \frac{|(x-\xi_j)\cdot\nabla v_i + 2t\partial_t v_i|^2}{|t|} \eta^2 G_{(\xi_k,0)}\, dxdt = 0, \tag{8.59}$$

for $0 \leq j \leq n-2$. Hence it is not hard to see that Fatou's Lemma and Fubini's theorem imply that for any $0 < t_1 < t_2 \leq 1$

$$\lim_{i\to\infty} \int_{-t_2^2}^{-t_1^2} \int_{B_1} \frac{|(x-\xi_j)\cdot\nabla v_i + 2t\partial_t v_i|^2}{|t|} = 0$$

for all $1 \leq j \leq n-2$. Therefore, one can deduce

$$\lim_{i\to\infty} \int_{-t_2^2}^{-t_1^2} \int_{B_1} \sum_{j=1}^{n-2} |\nabla_{y_j} v_i|^2 = 0 \ \text{ for any } 0 < t_1 < t_2 < 1. \tag{8.60}$$

Let

$$f_i(y) = \int_{(B_1^2\times\{y\})\times[-1,0]} |\partial_t u_i|^2((x,y),t)\, dxdt : B_1^{n-2} \subset \mathbb{R}^{n-2} \to \mathbb{R}_+,$$

and

$$g_i(y,t) = \int_{B_1^2} \sum_{j=1}^{n-2} |\nabla_{y_j} u_i|^2(x,y,t)\, dx : B_1^{n-2} \times [-1, -\frac{1}{8}] \to \mathbb{R}_+,$$

we define two local Hardy-Littlewood maximal functions of f_i on B_1^{n-2} and g_i on $B_1^{n-2} \times [-1, -\frac{1}{8}]$ as follows.

$$M(f_i)(y) = \sup_{0<r\leq 1} r^{2-n} \int_{B_r^{n-2}(y)} f_i \text{ for } y \in B_1^{n-2},$$

and

$$M(g_i)(y,t) = \sup_{0<r<1} r^{-n} \int_{B_r^{n-2}(y)\times[t-r^2,t]} g_i \ \text{ for } (y,t) \in B_1^{n-2} \times [-1, -\frac{1}{8}].$$

Then by (8.54) and (8.60), the weak L^1-estimate (see [192]) implies that there exist $y_i \in B^{n-2}_{\frac{1}{2}}$ and $t_i \in [-\frac{1}{2}, -\frac{1}{4}]$ such that

$$\lim_{i\to\infty} M(f_i)(y_i) = 0, \ \lim_{i\to\infty} M(g_i)(y_i, t_i) = 0. \tag{8.61}$$

Since v_i converges to constant strongly in $H^1_{\text{loc}}(P_2 \setminus (\mathbb{R}^{n-2} \times \mathbb{R}), \mathbb{R}^L)$, we can choose $x_i \in B^2_{\frac{1}{2}}$, and $\delta_i > 0$ such that

$$\begin{aligned}\delta_i^{-2} \int_{B^2_{\delta_i}(x_i)\times[t_i-\delta_i^2, t_i]} e(v_i)(x, y_i, t)\, dxdt &= \frac{\epsilon_0^2}{C(n)} \\ = \max\left\{ \delta_i^{-2} \int_{B^2_{\delta_i}(\tilde{x})\times[t_i-\delta_i^2, t_i]} e(v_i)(x, y_i, t)\, dxdt \mid \tilde{x} \in B^2_{\frac{1}{2}} \right\}.\end{aligned} \tag{8.62}$$

We claim that $\delta_i \to 0$ and $x_i \to 0 \in B^2_1$. For, otherwise, $\delta_i \geq \delta_0 > 0$. Then we choose $\phi \in C_0^\infty(B^2_{\delta_0})$ and compute, for any $y \in B^{n-2}_{\frac{1}{2}}(y_i)$ and $1 \leq j \leq n-2$,

$$\begin{aligned}
&\frac{\partial}{\partial y_j} \int_{t_i-\delta_0^2}^{t_i} \int_{\mathbb{R}^2} \phi^2(x) e(v_i)(x, y, t)\, dxdt \\
=\ & \int_{t_i-\delta_0^2}^{t_i} \int_{\mathbb{R}^2} \phi^2 \left[\langle \frac{\partial v_i}{\partial x}, \frac{\partial}{\partial x}(\frac{\partial v_i}{\partial y_j}) \rangle + \langle \frac{\partial v_i}{\partial y_l}, \frac{\partial}{\partial y_l}(\frac{\partial v_i}{\partial y_l}) \rangle + \langle \frac{f(v_i)}{\epsilon_i^2}, \frac{\partial v_i}{\partial y_j} \rangle \right] \\
=\ & -\int_{t_i-\delta_0^2}^{t_i} \int_{\mathbb{R}^2} \frac{\partial \phi^2}{\partial x} \cdot \langle \frac{\partial v_i}{\partial x}, \frac{\partial v_i}{\partial y_j} \rangle + \frac{\partial}{\partial y_l} \int_{t_i-\delta_0^2}^{t_0} \int_{\mathbb{R}^2} \phi^2 \langle \frac{\partial v_i}{\partial y_j}, \frac{\partial v_i}{\partial y_l} \rangle \\
& + \int_{t_i-\delta_0^2}^{t_0} \int_{\mathbb{R}^2} \phi^2 \langle -\Delta v_i + \frac{f(v_i)}{\epsilon_i^2}, \frac{\partial v_i}{\partial y_j} \rangle \\
=\ & -\int_{t_i-\delta_0^2}^{t_i} \int_{\mathbb{R}^2} \langle (\frac{\partial \phi^2}{\partial x} \cdot \frac{\partial v_i}{\partial x} + \phi^2 \frac{\partial v_i}{\partial t}), \frac{\partial v_i}{\partial y_j} \rangle \\
& + \frac{\partial}{\partial y_l} \int_{t_i-\delta_0^2}^{t_i} \int_{\mathbb{R}^2} \phi^2 \langle \frac{\partial v_i}{\partial y_j}, \frac{\partial v_i}{\partial y_l} \rangle.
\end{aligned} \tag{8.63}$$

Hence applying Lemma 4.2.10 with (8.61), we obtain

$$\delta_0^{-n} \int_{P_{\delta_0}((0, y_i), t_i)} e(v_i) \leq \frac{\epsilon_0^2}{2}$$

provided that $C(n)$ is chosen to be sufficiently large. This implies that $((0, y_i), t_0) \notin \Sigma_*$, which contradicts the structure of Σ_* we have obtained before. Hence $\delta_i \to 0$. It is easy to see that if $x_i \to x_0 \neq 0 \in B^2_1$, then $((x_0, y_i), t_i) \in \Sigma_*$, which is impossible again.

Now we perform another rescaling of v_i as follows. Define

$$\omega_i(x, y, t) = v_i((x_i, y_i, t_i) + (\delta_i x, \delta_i y), \delta_i^2 t)) \ \text{ on } \Omega_i,$$

where

$$\Omega_i \equiv \delta_i^{-1}(B^2_{\frac{1}{2}} \setminus \{x_i\}) \times (B_2^{n-2}) \times [-\frac{1}{2}\delta_i^{-2}, \frac{1}{2}\delta_i^{-2}] \to (\mathbb{R}^2 \times B_2^{n-2}) \times \mathbb{R}.$$

Then for $\hat{\epsilon_i} = \delta_i^{-1}\overline{\epsilon}_i$, ω_i satisfies

$$\partial_t \omega_i - \Delta \omega_i + \frac{f(\omega_i)}{\hat{\epsilon}_i^2} = 0 \ \text{ in } \Omega_i, \tag{8.64}$$

$$\begin{aligned} &\int_{B_1^2 \times (-1,0)} e(\omega_i)(x,0,t)\, dxdt = \frac{\epsilon_0^2}{C(n)} \\ &= \max\left\{ \int_{B_1^2(\tilde{x})\times(-1,0)} e(\omega_i)(x,0,t)\, dxdt \mid \tilde{x} \in \delta_i^{-1}(B^2_{\frac{1}{2}} \setminus \{x_i\}) \right\}, \end{aligned} \tag{8.65}$$

$$\lim_{i\to\infty} \sup_{r\in(0,\frac{1}{4\delta_i})} r^{2-n} \int_{B_r^{n-2}} \int_{B^2_{\frac{1}{2\delta_i}} \times[-\delta_i^{-2},0]} |\partial_t \omega_i|^2 = 0, \tag{8.66}$$

and

$$\lim_{i\to\infty} \sup_{r\in(0,\frac{1}{4\delta_i})} r^{-n} \int_{B_r^{n-2}\times[-r^2,0]} \int_{B^2_{\frac{1}{2\delta_i}}} \left(\sum_{j=1}^{n-2} |\nabla_{y_j} v_i|^2 \right) = 0. \tag{8.67}$$

Now we apply Lemma 4.2.10 twice to obtain a bubble. First, the same computation as (8.63) implies that for i sufficiently large

$$2^{2-n} \int_{(B_1^2(\tilde{x})\times B_2^{n-2})\times[-1,0]} e(\omega_i) \leq \frac{4\epsilon_0^2}{C(n)} \ \text{ for all } \tilde{x} \in \delta_i^{-1}(B^2_{\frac{1}{2}} \setminus \{x_i\}). \tag{8.68}$$

In particular, by using Fubini's theorem and (8.2.1), we can conclude that

$$\sup_{t\in[-\frac{1}{2},0]} 2^{2-n} \int_{B_2^n(\tilde{x},0)} e(\omega_i(t)) \leq \frac{8\epsilon_0^2}{C(n)}, \ \text{ for all } \tilde{x} \in \delta_i^{-1}(B^2_{\frac{1}{2}} \setminus \{x_i\}). \tag{8.69}$$

Next we compute that for any $s \in [-4, -\frac{1}{2}]$ and $t \in [-\frac{1}{2}, 0]$, $\phi \in C_0^\infty(B_2^n((\tilde{x},0)), \mathbb{R}_+)$ for $\tilde{x} \in \delta_i^{-1}(B^2_{\frac{1}{2}} \setminus \{x_i\})$,

$$\begin{aligned} &\left| \int_{\mathbb{R}^n} e(\omega_i(t))\phi - \int_{\mathbb{R}^n} e(\omega_i(s))\phi \right| \\ &\leq 2 \int_t^s \int_{B_2^n(\tilde{x},0)} |\partial_t \omega_i|^2 \phi + 2 \int_t^s \int_{B_2^n(\tilde{x},0)} |\nabla \omega_i| |\nabla \phi| |\partial_t \omega_i| \\ &\to 0 \ \text{ as } i \to \infty, \end{aligned}$$

where we have used (8.2.1) and (8.66). Therefore we obtain that for any $t \in [-4, -\frac{1}{2}]$,

$$(\frac{3}{2})^{2-n} \int_{B^n_{\frac{3}{2}}(\tilde{x},0)} e(\omega_i(t)) \leq \frac{16\epsilon_0^2}{C(n)}, \ \text{ for all } \tilde{x} \in \delta_i^{-1}(B^2_{\frac{1}{2}} \setminus \{x_i\}).$$

This, combined with (8.69), implies that for i sufficiently large

$$(\frac{3}{2})^{-n} \int_{P_{\frac{3}{2}}((\tilde{x},0),0)} e(\omega_i) \leq \frac{32\epsilon_0^2}{C(n)} \leq \frac{\epsilon_0^2}{2} \ \text{ for all } \tilde{x} \in \delta_i^{-1}(B^2_{\frac{1}{2}} \setminus \{x_i\}) \tag{8.70}$$

if $C(n)$ is chosen to be sufficiently large. Thus by Lemma 8.2.3 we have that for i sufficiently large

$$\max_{P_{\frac{5}{4}}((\tilde{x},0),0)} e(\omega_i) \leq C\epsilon_0 \text{ for all } \tilde{x} \in \delta_i^{-1}(B^2_{\frac{1}{2}} \setminus \{x_i\}). \tag{8.71}$$

From the local H^1 bound of ω_i in $\mathbb{R}^n \times \mathbb{R}_-$, we may assume that $\omega_i \to \omega_\infty$ weakly in $H^1_{\text{loc}}(\mathbb{R}^n \times \mathbb{R}_-, \mathbb{R}^L)$. It follows from (8.66) and (8.67) that

$$\int_{\mathbb{R}^n \times \mathbb{R}_-} \left(|\partial_t \omega_\infty|^2 + \sum_{j=1}^{n-2} |\frac{\partial \omega_\infty}{\partial y_j}|^2 \right) = 0.$$

Hence $\omega_\infty(x, y, t) = \omega_\infty(x) : \mathbb{R}^2 \to \mathbb{R}^L$ is independent of the last two variables y and t.

Moreover, by (8.71) we can assume that $\omega_i \to \omega_\infty$ in $C^2_{\text{loc}}((\mathbb{R}^2 \times B^{n-2}_{\frac{5}{4}}) \times [-(\frac{5}{4})^2, 0], \mathbb{R}^L)$. Thus by (8.65) we have

$$\frac{\epsilon_0^2}{C(n)} \leq \int_{\mathbb{R}^2} e(\omega_\infty) < +\infty.$$

As in §8.4, we know that either (i) $\hat{\epsilon}_i \to 0$, ω_∞ is a nontrivial harmonic map from $\mathbb{R}^2$ to N with finite energy and hence ω_∞ can be lifted as a bubble, or (ii) $\hat{\epsilon}_i \to \infty$, ω_∞ is a nontrivial harmonic function on $\mathbb{R}^2$ with finite energy which is impossible, or (iii) $\hat{\epsilon}_i \to c$ for some $0 < c < +\infty$ so that ω_∞ is a nontrivial solution to

$$\Delta \omega_\infty + \frac{f(\omega_\infty)}{c^2} = 0 \text{ in } \mathbb{R}^2$$

with $\int_{\mathbb{R}^2} e_c(\omega_\infty) < +\infty$, which is also impossible by Lemma 8.4.1.

Since N doesn't support harmonic S^2, the above argument shows that $u_i \to u$ strongly in $H^1_{\text{loc}}(M \times (0, T], N)$. Hence $\nu \equiv 0$ and $\mathcal{P}^n(\Sigma) = 0$. In particular, $\mathcal{P}^n(\text{sing}(u)) = 0$ (since $\text{sing}(u) \subset \Sigma$). From the strong convergence, we also know that u satisifes the three conditions (8.74), (8.75), and (8.76) below. Hence u is a suitable weak solution of (8.2). Thus we have now completed the proof of both Theorems 8.1.1 and 8.1.2 for $n \geq 3$. □

Before proving Theorem 8.1.5 and Theorem 8.1.6, we will prove a dimension reduction argument for weak solutions of the heat flow of harmonic maps that are obtained by Theorem 8.1.1 and Theorem 8.1.2, which can be viewed as the parabolic counterpart of Federer's dimension principle [54] (cf. also [171] and §2.3).

Theorem 8.5.2 *For $n \geq 3$, assume that N doesn't support any harmonic S^2. Let $u \in H^1(M \times (0, T), N)$ be a weak solution of (8.2) obtained by either Theorem 8.1.1 or Theorem 8.1.2. Then $\mathcal{P}^n(\text{sing}(u)) = 0$. Furthermore, sing$(u)$ is discrete for $n = 3$. For $n \geq 4$, $\dim_{\mathcal{P}}(\text{sing}(u)) \leq n - 2$; and if in addition N doesn't support harmonic S^3 then $\dim_{\mathcal{P}}(\text{sing}(u)) \leq n - 3$.*

In order to prove this theorem, we need a simple lemma.

Lemma 8.5.3 *Suppose that $f \in H^1_{loc}(\mathbb{R}^{n+1}_-)$ is such that for some $z_0 = (x_0, t_0) \neq (0,0) \in \mathbb{R}^{n+1}_-$,*

$$x \cdot \nabla f + 2t\frac{\partial f}{\partial t} = (x - x_0) \cdot \nabla f + 2(t - t_0)\frac{\partial f}{\partial t} = 0 \text{ a.e. } (x,t) \in \mathbb{R}^{n+1}_-. \tag{8.72}$$

Then either (i) for $t_0 = 0$ $f(x + x_0, t) = f(x,t)$ for a.e. $(x,t) \in \mathbb{R}^{n+1}_-$; or (ii) for $t_0 \neq 0$ $f_t(x,t) = x_0 \cdot \nabla f(x,t) = 0$ for a.e. $(x,t) \in \mathbb{R}^{n+1}_-$. In particular, $f(x,t) = f(x)$ is independent of both t variable and x_0-direction, and is homogeneous of degree zero as well.

Proof. If $t_0 = 0$, then it is easy to see that (8.72) implies that $x_0 \cdot \nabla f(x,t) = 0$ a.e. $(x,t) \in \mathbb{R}^{n+1}_-$. Hence f is independent of x_0-direction. For $t_0 \neq 0$, we define $f_0(x,t) = f(x_0 + x, t_0 + t)$ for $(x,t) \in \mathbb{R}^{n+1}_-$. Then (8.72) implies that for any $R > 0$ $f_0(Rx, R^2t) = f_0(x,t)$ a.e. $(x,t) \in \mathbb{R}^{n+1}_-$, and

$$x_0 \cdot \nabla f_0 + 2t_0\frac{\partial f_0}{\partial t} = 0 \text{ a.e. } (x,t) \in \mathbb{R}^{n+1}_-.$$

Replacing (x,t) by (Rx, R^2t) and using the self-similarity of f_0, we obtain that for any $R > 0$

$$x_0 \cdot \nabla f_0 + 2Rt_0\frac{\partial f_0}{\partial t} = 0 \text{ a.e. } (x,t) \in \mathbb{R}^{n+1}_-.$$

Sending R to zero gives

$$x_0 \cdot \nabla f_0 = \frac{\partial f_0}{\partial t} = 0 \text{ a.e. } (x,t) \in \mathbb{R}^{n+1}_-.$$

This proves (ii). □

Proof of Theorem 8.5.2:
Let's first prove that $\text{sing}(u)$ is discrete for $n = 3$. If $\text{sing}(u)$ is not discrete, then there are $z_i \in \text{sing}(u)$ with $z_i \to z_0 \in \text{sing}(u)$. Therefore we define $\lambda_i = 4\delta(z_i, z_0)$ and consider the scaled maps $u_{\lambda_i}(x,t) = u(z_0 + (\lambda_i x, \lambda_i^2 t))$. By the bound on $\Theta^n(u, z_0)$, we can assume that u_{λ_i} converges strongly in $H^1_{\text{loc}}(P_1, N)$ to a self-similar solution $u_0 : \mathbb{R}^3_- \to N$ (i.e. $x \cdot \nabla u_0 + 2t\frac{\partial u_0}{\partial t} = 0$ a.e. $(x,t) \in \mathbb{R}^3_-$). Moreover, both $(0,0)$ and a point $z_1 = (x_1, t_1) \in \partial P_{\frac{1}{4}}$ are contained in $\text{sing}(u_0)$. By blowing up u_0 and z_1 one more time, we would obtain another weak solution $u_1 \in H^1(\mathbb{R}^3_-, N)$ of (8.2), which is self-similar with respect to both $(0,0)$ and z_1. Applying Lemma 8.72 would then give us that either $u_1(x_1, x_2, x_3, t) = u_1(x_1, x_2, x_3)$ is a harmonic S^2 or a quasi-harmonic S^2. Both of them are impossible. Hence $\text{sing}(u)$ is at most discrete.

For $n \geq 4$, we argue as follows. First note that $\text{sing}(u)$ is given by

$$\text{sing}(u) = \left\{ (x,t) \in M \times (0,T) : \lim_{r \downarrow 0} r^{-n} \int_{P_r(x,t)} |\nabla u|^2 \geq \epsilon_0^2 \right\},$$

where ϵ_0 is given by (8.2.3). It follows from $\mathcal{P}^n(\text{sing}(u)) = 0$ that $\dim_{\mathcal{P}}(\text{sing}(u)) \leq n$. Let $0 \leq s < n$ be such that $\mathcal{P}^s(\text{sing}(u)) > 0$. Then there exists $z_0 \in \text{sing}(u)$ such that (cf. [55])

$$\lim_{r_i \downarrow 0} r_i^{-s} \mathcal{P}^s(\text{sing}(u) \cap P_{r_i}(z_0)) > 0. \tag{8.73}$$

Let $u_i(x,t) = u(z_0 + (r_i x, r_i^2 t)) : P_1 \to N$. Then we have

$$\sup_{i\geq 1} \int_{P_1} \left(|\partial_t u_i|^2 + |\nabla u_i|^2\right) < \infty$$

so that we may assume u_i converges u_0 weakly in $H^1(P_1, N)$, which is strong in $H^1_{\rm loc}(P_1, N)$ by Theorem 8.1.1. Hence u_0 is a weak solution of (8.2) with

$$\int_{T_r} |2t\partial_t u_0 + x\cdot\nabla u_0|^2 = 0 \text{ for all } r > 0.$$

Therefore either $u_0(x,t) = u_0(\frac{x}{|x|}) : \mathbb{R}^n \to N$ (i.e., u_0 is a homogeneous of degree zero harmonic map from $\mathbb{R}^n$ to N) or $u_0(x,t) = u_0(\frac{x}{\sqrt{-t}}) : \mathbb{R}^{n+1}_- \to N$ is a self-similar solution of (8.2). In the former case, by the result of Lin [123] in §4, we have $\dim_{\mathcal{P}}(\text{sing}(u_0)) \leq n-3$ and we are done.

Next we will only consider the latter case at each following step. Note that $\text{sing}(u_i) \cap P_{\frac{1}{2}} = \mathcal{P}_{z_0,r_i}(\text{sing}(u_i) \cap P_{\frac{r_i}{2}}(z_0))$ so that (8.73) implies

$$\lim_{i\to\infty} \mathcal{P}^s(\text{sing}(u_i) \cap P_{\frac{1}{2}}) > 0.$$

Therefore by the small energy regularity theorem we can conclude that

$$\mathcal{P}^s(\text{sing}(u_0) \cap P_{\frac{1}{2}}) > 0.$$

There are two possibilities: either (i) $s \leq 0$, or (ii) $s > 0$ and there is $z_1 = (x_1, t_1) \in \text{sing}(u_0) \cap \partial P_1$ such that

$$\limsup_{r\to 0} r^{-s}\mathcal{P}^s(\text{sing}(u_0) \cap P_r(z_1)) > 0.$$

Repeating the above blow up argument of u_0 at z_1, we would have get another weak solution of $u_1 \in H^1_{\rm loc}(\mathbb{R}^{n+1}_-, N)$ of (8.2) with $\mathcal{P}^s(\text{sing}(u_1) \cap P_1) > 0$. From our assumption that there is no static tangent map at z_1, this and Lemma 8.5.3) would imply that u_1 is independent of x_1 direction, i.e., $u_1((x_1, y, t)) = u_1(\frac{y}{\sqrt{-t}})$ for any $(x_1, y, t) \in \mathbb{R}^{n+1}_-$. If $s - 1 \leq 0$, we stop. Otherwise, there is a point $z_2 \in \text{sing}(u_1) \cap \big(\partial P_1 \cap ((\{0\}\mathbb{R}^{n-1}) \times \mathbb{R}_-)\big)$ with

$$\lim_{r\downarrow} r^{-s}\mathcal{P}^s(\text{sing}(u_1) \cap P_r(z_2)) > 0.$$

We repeat the same argument at z_2. If we repeat the procedure m times, we would get a weak solution $u_m \in H^1_{\rm loc}(\mathbb{R}^{n+1}_-, N)$ which is a self-similar solution of (8.2) and satisfies $u_n(x_1, \cdots, x_m, y, t) = u_n(\frac{y}{\sqrt{-t}})$ for any $(x_1, \cdots, x_m, y) \in \mathbb{R}^m \times \mathbb{R}^{n-m} = \mathbb{R}^n$ and $\mathcal{P}^s(\text{sing}(u_m) \cap P_1) > 0$. We can repeat the argument until $s - m \leq 0$. In order to have constructed u_m, we must have $s - m + 1 > 0$. Since $s < n$ and m is integer we then have $m \leq n-1$. If $m \geq n-2$, then we would have a map $u_m : \mathbb{R}^n \times \mathbb{R}_- \to N$ such that $\mathbb{R}^{n-2} \times R(t) \subset \Sigma_n$, here $R(t) \subset \mathbb{R}^2 \times \mathbb{R}_-$ is a self-similar curve passing through 0 and $R(t) \neq \{0\}$. Hence $\mathcal{P}^n(\Sigma_m) > 0$, which is impossible. Therefore

$m \leq n-3$. □

Completion of proof of Corollaries 8.1.5, 8.1.6 and Theorem 8.1.7:

Applying the above blow up argument, it is not hard to see that if $\Sigma \neq \emptyset$, then we would obtain either a harmonic S^k for some $2 \leq k \leq n-1$ or a quasi-harmonic S^l for some $3 \leq l \leq n$, which would contradict our assumption. Hence $\Sigma = \emptyset$ and the convergence is in $C^2(M \times (0,T], N)$ and the solution $u \in C^\infty(M \times (0,T], N)$. Moreover, the global gradient estimate (8.4) holds. □

To conclude this section, we introduce the notion of suitable weak solutions of (8.2).

Definition 8.5.4 A weak solution $u : M \times (0,T] \to N$ of (8.2) is called a *suitable* weak heat flow of harmonic maps, if u satisfies the following three properties:
(i) the energy inequality: for any $\phi \in C_0^\infty(M)$ and $0 < t_1 \leq t_2 \leq T$,

$$\begin{aligned} &\int_M |\nabla u(t_2)|^2 \phi^2 + 2\int_{t_1}^{t_2}\int_M |\partial_t u|^2 \phi^2 \\ &\leq \int_M |\nabla u(t_1)|^2 \phi^2 - 2\int_{t_1}^{t_2}\int_M \langle \partial_t u, \nabla u\rangle \cdot \nabla \phi^2. \end{aligned} \tag{8.74}$$

(ii) the energy monotonicity inequality: for any $z_0 = (x_0, t_0) \in M \times (0,T)$ and $0 < R_1 \leq R_2 \leq \min\{\sqrt{t_0}, i_M\}$

$$\begin{aligned} &c\int_{R_1}^{R_2} \frac{1}{r}\int_{T_r(z_0)} \eta^2 \frac{|(x-x_0)\cdot\nabla u + 2(t-t_0)\partial_t u|^2}{|t_0 - t|} G_{z_0} \\ &\leq e^{C(R_2-R_1)}\Psi(u, z_0, R_2) - \Psi(u, z_0, R_1) + C(R_2 - R_1). \end{aligned} \tag{8.75}$$

(iii) ϵ_0-regularity: there exists $\epsilon_0 > 0$, $\delta_0 > 0$, and $C_0 > 0$ such that

$$\Psi(u, z_0, R) \leq \epsilon_0^2 \Rightarrow \sup_{P_{\delta_0 R}(z_0)} |\nabla u|^2 \leq C(\epsilon_0, R). \tag{8.76}$$

We would like to remark that our blow-up analysis outlined in §8.4 and §8.5 above can be applied to the class of *suitable* weak heat flows of harmonic maps, since the above three properties are all what we need for the blow up analysis.

In particular, one has the following corresponding theorem of Theorem 8.1.1 and Theorem 8.1.5 for suitable weak solutions to the heat flow of harmonic maps.

Theorem 8.5.5 *Assume that N doesn't support any harmonic S^2. Let $\{u_i\} \subset H^1(M \times (0,T), N)$ solve (8.2) weakly and satisfy*

$$\sup_i \int_{M\times[0,T]} \left(|\partial_t u_i|^2 + |\nabla u_i|^2\right) = K < \infty.$$

If, in addition, u_i, $i \geq 1$, is a suitable weak heat flow of harmonic maps. Then u_i (after passing to possible subsequences) converges strongly in $H^1_{loc}(M \times (0,T], N)$ to $u \in H^1(M \times [0,T], N)$, which is also a suitable weak heat flow of harmonic map. In particular, $\mathcal{P}^n(sing(u)) = 0$. Furthermore, if N doesn't support neither harmonic S^k for $2 \leq k \leq n-1$ nor quasi-harmonic S^l for $3 \leq l \leq n$, then $u \in C^\infty(M \times (0,T], N)$.

It follows from a simple energy consideration that there doesn't exist any quasi-harmonic S^2 in any compact Riemannian manifold N. It also follows from §8.4 that (8.75) is not needed in the blow up analysis in dimensions two. Moreover, in dimensions two (8.76) is equivalent to

$$\sup_{t\in[t_0-R^2,t_0]}\int_{B_R(x_0)}|\nabla u|^2\le\epsilon_0^2\Rightarrow\sup_{P_{\delta_0 R}(z_0)}|\nabla u|^2\le C(\epsilon_0,R), \tag{8.77}$$

which holds for any weak solution of (8.2) in dimensions two (see Wang [208] for a proof). Moreover, in dimension two (8.74) can be replaced by a weaker version: for any $\phi\in C_0^\infty(M)$ and $0<t_1\le t_2\le T$,

$$\left|\int_{M^2}|\nabla u(t_2)|^2\phi^2-\int_{M^2}|\nabla u(t_1)|^2\phi^2\right|\le C(\phi,E_0)\,|t_2-t_1|^{\frac{1}{2}}. \tag{8.78}$$

Therefore we have an immediate corollary

Corollary 8.5.6 *For $n=2$, assume that N doesn't support any harmonic S^2. Let $u\in H^1(M^2\times(0,T],N)$ be a weak solution of (8.2), which satisfies the energy inequality (8.78). Then $u\in C^\infty(M^2\times(0,T],N)$. Moreover*

$$\sup_{M^2\times(t_0,T]}|\nabla u|^2\le C(E_0,t_0)\quad \textit{for any } t_0>0.$$

Remark 8.5.7 Note first that according to our definition that any suitable weak heat flow of harmonic map enjoys the partial regularity property, i.e. $\mathcal{P}^n(\text{sing}(u))=0$. We would like to remark that a weak solution satisfying (i) and (ii) of the definition of *suitable* weak heat flow of harmonic maps has been introduced by Feldman [58] and Chen-Li-Lin [29]. Moreover, it was proved by [58] and [29] that when $N=S^{L-1}\subset\mathbb{R}^L$ any weak solution of (8.2) satisfying (i) and (ii) enjoys the partial regularity property. Chen-Wang [34] have extended [58] and [29] to any Riemannian homogeneous manifold. For any compact Riemannian manifold N, Moser [148] has proved the partial regularity theorem for $n\le 4$.

Chapter 9

Dynamics of defect measures in heat flows

In this chapter, we will continue the blow-up analysis, with the emphasis on the dynamic behavior of the defect measures, for weakly convergent sequence of smooth solutions (or certain classes of weak solutions to be specified later) to the heat flow of harmonic maps or approximate harmonic maps (i.e., critical points of the Ginzburg-Landau energy functional), which we started in §7.

A general situation for the heat flow of harmonic maps is as follows: For a bounded domain $\Omega \subseteq \mathbb{R}^n$, let $u_i : \Omega \times \mathbb{R}_+ \to N$ be smooth solutions to the heat flow of harmonic maps from Ω to N (8.2) on $\Omega \times \mathbb{R}_+$ such that u_i weakly converges to u in $H^1_{\text{loc}}(\Omega \times \mathbb{R}_+, N)$. By Fatou's lemma, we assume that

$$\frac{1}{2}|\nabla u_i|^2(x,t)\,dxdt \to \frac{1}{2}|\nabla u|^2(x,t)\,dxdt + \nu \tag{9.1}$$

and

$$|\partial_t u_i|^2(x,t)\,dxdt \to |\partial_t u|^2(x,t)\,dxdt + \eta \tag{9.2}$$

as convergence of Radon measures on $\Omega \times \mathbb{R}_+$ for two nonnegative Randon measures ν, η supported on the concentration set Σ defined by (8.17). By Lemma 8.2.9 we have $\nu = \nu_t\,dt$ for some nonnegative Radon measures $\{\nu_t\}_{t\in\mathbb{R}_+}$ in Ω.

By the analysis of §8, we know that the nonexistence of harmonic S^2 in N is (essentially) necessary and sufficient condition for such weakly convergent sequences to be strongly convergent. The blow up analysis in §8 also indicates that if N does admit harmonic S^2 and the strong convergence fails, then the concentration set $\Sigma = \cup_{t>0}\Sigma_t$ has positive, locally finite n-dimensional parabolic Hausdorff measure and the n-dimensional density of ν, $\Theta^n(\mu, \cdot)$, is positive and finite. Moreover, for L^1 a.e. $t \in \mathbb{R}_+$, the euclidean $(n-2)$ density of ν_t, $\Theta^{n-2}(\nu_t, \cdot)$, is positive and finite for H^{m-2} a.e. $x \in \Sigma_t$.

In this chapter, we will show that for L^1 a.e. $t \in \mathbb{R}_+$, both Σ_t and ν_t are $(n-2)$-rectifiable. To prove this, we develop a generalized varifold approach which is an extension of the classical varifold approach by Almgren [5] and Allard [2]. Roughly speaking, we associate u_i with a $(n-2)$-generalized varifold V_{u_i} on $\Omega \times \mathbb{R}_+$ and show that V_{u_i} converges to a $(n-2)$-generalized varifold $V = V_t\,dt$, and V_t

has a generalized mean curvature $H_t \in L^2_{\|V_t\|}(\Omega, \mathbb{R}^n)$ for L^1 a.e. $t \in R_+$, and then employing an extension of Allard's rectifiability theorem for classical varifolds to show that $V_t\lfloor(\{x \in \Omega \mid \Theta^{n-2}(\|V_t\|, x) > 0\})$ is $(n-2)$-rectifiable.

Another major theorem of this chapter is that $(u, \nu_t\, dt)$ satisfies a generalized varifold flow, which reduces to Brakke's motion of varifolds by mean curvature if u is a weak heat flow of harmonic map (e.g. $u \in C^\infty(\Omega \times \mathbb{R}_+, N)$) such that (8.74) becomes an equality, i.e.,

$$\overline{\nabla}_t \nu_t(\phi) = \limsup_{s \to t} \frac{\nu_s(\phi) - \nu_t(\phi)}{s - t} \leq -\int_\Omega \left(\phi |H_t|^2 - \langle (T_x \Sigma_t)^\perp \nabla \phi, H_t \rangle\right) d\nu_t \qquad (9.3)$$

for any $t \geq 0$ and $\phi \in C_0^2(\Omega, \mathbb{R}_+)$. See Brakke [15] and Ilmanen [99, 100] for more details. We also would like to mention the interesting papers by Ambrosio-Soner [7, 8] for further discussions on mean curvature flows. We would also like to remark that a slightly weaker version of the above fact has also been obtained by Li-Tian [137] independently. In this Chapter, we also establish an energy quantization result for the density function $\Theta^{n-2}(\nu_t, \cdot)$ on the concentration set Σ_t for both heat flows of harmonic maps and Ginzburg-Landau heat flows into S^{L-1}. Such a quantization has been previously obtained by Lin-Riviere [131] for stationary harmonic maps, and Lin-Wang [135] for critical points of the Ginzburg-Landau functional.

Since we can treat smooth heat flows of harmonic maps in a slightly simpler way, we still decide to work on Ginzburg-Landau heat flows. Also, we will continue to use the notations developed from both Chapter 7 and 8.

This chapter is organized as follows. In §9.1, we have a brief tour of generalized varifolds and prove the rectifiability of Σ_t for L^1 a.e. t. In §9.2, we present the motion law for the generalized defect varifolds $(u, \nu_t\, dt)$. In §9.3, we prove the energy quantization result.

9.1 Generalized varifolds and rectifiability

In this section, we first recall some of the basic theory of classical varifolds, developed by Almgren [4, 5] and Allard [2] (see also Simon [189]), and the notion of generalized varifolds, which was remarked by Almgren [5], recently explored by Ambrosio-Soner [7] in the study of the dynamics of Ginzburg-Landau equations with complex values and by Lin [124] in the study of mapping problems.

For $1 \leq l \leq n$, let $G_l(n)$ denote the Grassmann manifold of l-dimensional un oriented planes in $\mathbb{R}^n$. Recall a l-varifold in a bounded domain $\Omega \subseteq \mathbb{R}^n$ is a Radon measure in $\Omega \times G_l(n)$. Let $V_l(\Omega)$ denote the set of all l-varifolds in Ω. The weight $\|V\|$ of $V \in V_l(\Omega)$ is a Randon measure in Ω defined by $\|V\| \equiv \pi_\#(V)$, where $\pi(x, A) = x : \Omega \times G_l(n) \to \Omega$ is projection to the first argument.

$V \in V_l(\Omega)$ is said to be a l-rectifiable varifold if there exist a l-rectifiable set $E \subset \Omega$ and a locally H^l-integrable, positive function θ such that

$$V = \delta_{T_x E} \theta H^l \lfloor E \ \text{ for } H^l \ \text{ a.e. in } \Omega$$

where $T_x E$ denotes the tangent plane of E at x and $\delta_{T_x E}$ denotes the Dirac mass at $T_x E \subseteq G_l(n)$. Let $RV_l(\Omega)$ denote the set of all l-rectifiable varifolds in Ω.

We now recall the definition of generalized varifolds (see [7]).

Definition 9.1.1 A l-dimensional generalized varifold V in Ω is a nonnegative Radon measure on $\Omega \times A_{l,n}$, where

$$A_{l,n} = \left\{A \in \mathbb{R}^{n\times n} \mid A \text{ is symmetric and } \operatorname{tr}(A) = l\ , -lI_n \leq A \leq I_n\right\} \tag{9.4}$$

where $\operatorname{tr}(A)$ is the trace of A and $I_n \in \mathbb{R}^{n\times n}$ is the identity matrix of order n. The class of all generalized l-varifolds in Ω is denoted by $V_l^*(\Omega)$. Let $\|V\|$ denote the weight of $v \in V_l^*(\Omega)$.

Since $G_l(n) \subset A_{l,n}$, it is clear that $V_l(\Omega) \subset V_l^*(\Omega)$. We now introduce the notion of the first variation of generalized varifolds.

Definition 9.1.2 For any $V \in V_l^*(\Omega)$, the first variation of V, δV, is a distribution on $C_0^1(\Omega, \mathbb{R}^n)$ defined by

$$\delta V(X) = -\int_{\Omega\times A_{l,n}} \nabla X(x) : A dV(x, A),\ \forall X \in C_0^1(\Omega, R^n), \tag{9.5}$$

where : is a scalar product on $\mathbb{R}^{n\times n}$ defined by

$$A : B = \sum_{ij} A_{ij}B_{ij} \text{ for } A,\ B \in \mathbb{R}^{n\times n}.$$

V is called a stationary l-varifold if $\delta V \equiv 0$.

Note that if δV is a Radon measure, i.e.,

$$\begin{aligned} \|\delta V\|(G) &= \sup\left\{|\delta V(X)| \mid X \in C_0^1(\Omega, \mathbb{R}^n),\ \|X\|_{L^\infty} \leq 1,\ \operatorname{supp}(X) \subset G\right\} \\ &\leq C(G) < \infty,\ \forall G \subset\subset \Omega, \end{aligned}$$

then by the Riesz representation theorem there exists a $\|\delta V\|$-measurable, S^{n-1}-valued function β such that

$$\delta V(X) = \int_\Omega X(x)\beta(x)\, d\|\delta V\|(x),\ \forall X \in C_0^1(\Omega, \mathbb{R}^n). \tag{9.6}$$

If, in additions, $\|\delta V\|$ is absolutely continuous with respect to $\|V\|$, then there exists a $\|V\|$-measurable function $H : \Omega \to \mathbb{R}^n$ such that

$$\delta V(X) = \int_\Omega \langle H(x), X(x)\rangle\, d\|V\|(x),\ \forall X \in C_0^1(\Omega, \mathbb{R}^n). \tag{9.7}$$

Such a function H is called to be *generalized mean curvature* of V.

Recall also that a sequence of generalized varifolds $\{V_i\} \subset V_l^*(\Omega)$ converges to $V \in V_l^*(\Omega)$ if $V_i \to V$, as convergence of Radon measures on $\Omega \times A_{l,n}$. Therefore if $V_i \to V$, then we also have $\delta V_i \to \delta V$ in the sense of distributions. In particular, if for $A \subset\subset \Omega$, $\sup_i \|\delta V_i\|(A) < \infty$, then by the lower semicontinuity we have

$$\|\delta V\|(A) \leq \liminf_{i\to\infty} \|\delta V_i\|(A) < \infty. \tag{9.8}$$

To motivate the application of generalized varifolds to the problem we are studying in this chapter, we give two examples.

Example 9.1.3 For $u \in H^1(\Omega, R^L)$, define

$$V_u(x) = \frac{1}{2}\delta_{A(u)(x)} \left|\nabla u\right|^2(x)\,dx,$$

where $\delta_{A(u)(x)}$ is Dirac mass at $A(u)(x) \in A_{n-2,n}$, which is defined by

$$A(u)(x) = \begin{cases} I_n - 2\frac{\nabla u \otimes \nabla u}{|\nabla u|^2}(x), & \text{if } |\nabla u|(x) \neq 0, \\ I_{n-2}, & \text{if } |\nabla u|(x) = 0. \end{cases} \tag{9.9}$$

Then we have $V_u \in V^*_{n-2}(\Omega)$. Moreover, for any Borel set $B \subset \Omega \times A_{n-2,n}$,

$$V_u(B) = \frac{1}{2}\int_{\pi(B)} |\nabla u|^2$$

where $\pi(B) = \{x \in \Omega \mid (x, A_u(x)) \in B\}$. It is readily seen that for any $X \in C^1_0(\Omega, \mathbb{R}^n)$,

$$\begin{aligned} \delta V_u(X) &= -\int_\Omega \nabla X(x) : A\, dV_u(x, A) \\ &= -\frac{1}{2}\int_\Omega \nabla X(x) : A(u)(x)|\nabla u|^2(x) \\ &= -\frac{1}{2}\int_\Omega \left(|\nabla u|^2 \mathrm{div}(X) - 2\nabla_i u \nabla_j u \nabla_i X^j\right). \end{aligned}$$

If $u \in H^1(\Omega, N)$ is a stationary harmonic map, then by (3.41) we have

$$\delta V_u(X) = 0, \ \forall X \in C^1_0(\Omega, \mathbb{R}^n),$$

so that V_u is a stationary generalized $(n-2)$-varifold in Ω.

Example 9.1.4 For $\epsilon_i \downarrow 0$, let $u_i \in H^1(\Omega, R^L)$ be critical points of $E_{\epsilon_i}(\cdot)$, i.e.,

$$\Delta u_i + \frac{1}{\epsilon_i^2} f(u_i) = 0 \ \text{ in } \Omega. \tag{9.10}$$

Then it follows from the discussion of §7 that for any $X \in C^1_0(\Omega, \mathbb{R}^n)$,

$$\delta V_{u_{\epsilon_i}}(X) = -\int_\Omega \frac{F(u_i)}{\epsilon_i^2} \mathrm{div} X. \tag{9.11}$$

If $\sup_i E_{\epsilon_i}(u_i) < \infty$, then we may assume that $u_i \to u$ weakly in $H^1(\Omega, R^L)$, and

$$e(u_i)(x)\,dx \to \frac{1}{2}|\nabla u|^2(x)\,dx + \nu$$

as convergence of Radon measures in Ω for some nonnegative Radon measure ν on Ω, and there exists $V \in V^*_{n-2}(\Omega)$ such that V_{u_i} weakly converges to V. Moreover, Lemma 8.21 gives

$$\lim_{i \to \infty} \int_\Omega \frac{F(u_i)}{\epsilon_i^2} = 0$$

so that

$$\delta V(X) = \lim_{i\to\infty} \delta V_{u_i}(X) = 0, \ \forall X \in C_0^1(\Omega, \mathbb{R}^n)$$

and hence V is stationary. From the discussion in [134], we know that $\nu = \theta H^{n-2}\lfloor\Sigma$ is a $(n-2)$-rectifiable generalized varifold. Later in this section, we will see that $V = V_u + V_\nu$, where V_ν is the $(n-2)$-rectifiable varifold associated with ν and is given by

$$V_\nu = \delta_{T_x\Sigma}\theta H^{n-2}\lfloor\Sigma,$$

where Σ is the concentration set associated with u_i.

We now start to apply the generalized varifold approach to the study of Ginzburg-Landau heat flows. For any $\epsilon_i \downarrow 0$, let $u_i \in C^\infty(\Omega \times \mathbb{R}_+, \mathbb{R}^L)$ solve

$$\partial_t u_i - \Delta u_i = \frac{1}{\epsilon_i^2} f(u_i) \text{ in } \Omega \times \mathbb{R}_+. \tag{9.12}$$

Assume that

$$\sup_i \sup_{0<t<\infty} \left(\int_0^t \int_\Omega |\partial_t u_i|^2 + \int_\Omega e(u_i)(x,t)\, dx \right) \equiv K < +\infty. \tag{9.13}$$

For each u_i, associate $V_i \in V_{n-2}^*(\Omega \times \mathbb{R}_+)$ as follows:

$$V_i(x,t,A) = \delta_{A(u_i)(x,t)}(A)\mu_t^i\, dt, \ \forall (x,t,A) \in \Omega \times R_+ \times A_{n-2,n},$$

where $A(u_i)$ is defined by (9.9) and $\mu_t^i(x) = e_{\epsilon_i}(u_i)(x,t)\, dx$.

Let $\pi : \Omega \times \mathbb{R}_+ \times A_{n-2,n} \to \Omega \times \mathbb{R}_+$ be the projection map. Then the weight $\|V_i\| = \pi_\#(V_i) = \mu_t^i\, dt$ and hence $\sup_i \|V_i\|(G) < \infty$ for any compact subset $G \subset\subset \Omega \times \mathbb{R}_+$. Therefore we may assume that there exists $V \in V_{n-2}^*(\Omega \times \mathbb{R}_+)$ such that

$$V_i \to V, \quad \|V_i\| = \mu_t^i\, dt \to \|V\| \tag{9.14}$$

as convergence of Radon measures on $\Omega \times \mathbb{R}^+$. By Lemma 8.2.9, $\|V\| = \mu_t\, dt$ for some nonnegative Radon measures $\{\mu_t\}_{t>0}$ on Ω. For each $(x,t) \in \Omega \times \mathbb{R}_+$, there is a probability measure $V_{x,t}$ on $A_{n-2,n}$ such that

$$V = V_{x,t}\|V\| = V_{x,t}\mu_t\, dt.$$

Note that for any compact subset $G \subset\subset \Omega \times \mathbb{R}_+$

$$\|\nabla u_i \cdot \partial_t u_i\|_{L^1(G)}$$

is uniformly bounded. Hence we may assume that there is a (signed) Radon measure σ on $\Omega \times \mathbb{R}_+$ such that

$$-\nabla u_i \cdot \partial_t u_i\, dxdt \to \sigma$$

as convergence of Radon measures on $\Omega \times \mathbb{R}_+$. Since $-\nabla u_i \cdot \partial_t u_i\, dxdt$ is absolutely continuous with respect to $e(u_i)(x,t)\, dxdt$, σ is absolutely continuous with respect to $\|V\|$. Hence by the Riesz representation theorem again, there exists a $H_t \in L^1_{\|V\|}(\Omega \times \mathbb{R}, \mathbb{R}^n)$ such that

$$\sigma(x,t) = H_t(x)\mu_t(x)\, dt. \tag{9.15}$$

Moreover, by the lower semicontinuity, we have

$$\begin{aligned}\int_0^\infty \int_\Omega |H_t(x)|^2 \, d\mu(x,t) &\leq \liminf_{i\to\infty} \int_0^\infty \int_\Omega \left| \frac{\nabla u_i \cdot \partial_t u_i}{e(u_i)} \right|^2 e(u_i) \\ &\leq 2 \liminf_{i\to\infty} \int_0^\infty \int_\Omega |\partial_t u_i|^2 < \infty, \qquad (9.16)\end{aligned}$$

where we have used the Cauchy-Schwarz inequality in the last step.

Lemma 9.1.5 *For L^1 a.e. $t \in \mathbb{R}_+$, $V_t = V_{x,t}\mu_t \in V^*_{n-2}(\Omega)$ has its first variation δV_t absolutely continuous with respect to μ_t. Moreover $\delta V_t = H_t \mu_t$ with $H_t \in L^2_{\mu_t}(\Omega, \mathbb{R}^n)$.*

Proof. For $Y \in C^1_0(\Omega, \mathbb{R}^n)$ and $\gamma \in C_0(\mathbb{R}_+, \mathbb{R})$, denote $V^i_t = V_{A(u_i)(x,t)}\mu^i_t \in V^*_{n-2}(\Omega)$. Then we have

$$\begin{aligned}\int_{\mathbb{R}_+} \gamma(t) \delta V^i_t(Y) \, dt &= -\int_{\mathbb{R}_+} \gamma(t) \int_\Omega \nabla Y : A \, dV^i_t(x, A) \\ &= -\frac{1}{2} \int_{\mathbb{R}_+} \gamma(t) \int_\Omega \nabla Y : \left(|\nabla u_i|^2 I_n - 2\nabla u_i \otimes \nabla u_i \right) \\ &\quad - \int_{\Omega\times\mathbb{R}_+} \gamma(t) \nabla Y : A(u_i) \frac{F(u_i)}{\epsilon_i^2} \\ &= I + II\end{aligned}$$

For I, multiplying (9.12) by $Y(x) \cdot \nabla u_i$ and using integration it by parts, we have

$$\begin{aligned}&\int_{\mathbb{R}_+} \gamma(t) \int_\Omega \langle Y(x) \cdot \nabla u_i, \partial_t u_i \rangle \\ &= \int_{\mathbb{R}_+} \gamma(t) \int_\Omega \left(\Delta u_i + \frac{1}{\epsilon_i^2} f(u_i) \right) Y(x) \cdot \nabla u_i \\ &= \int_{\mathbb{R}_+} \gamma(t) \int_\Omega \langle \Delta u_i, Y(x) \cdot \nabla u_i \rangle + \int_{\mathbb{R}_+} \gamma(t) \int_\Omega \nabla \left(\frac{F(u_i)}{\epsilon_i^2} \right) \cdot Y(x) \\ &= \int_{\mathbb{R}_+} \gamma(t) \int_\Omega \left[(\nabla_j u_i \nabla_l u_i Y^l)_j - Y \cdot \nabla \left(\frac{|\nabla u_i|^2}{2}\right) - \nabla_j u_i \nabla_l u_i Y^l_j \right] \\ &\quad + \int_{\mathbb{R}_+} \gamma(t) \int_\Omega \frac{F(u_i)}{\epsilon_i^2} \mathrm{div} Y \\ &= \int_{\mathbb{R}_+} \gamma(t) \int_\Omega \left(\frac{1}{2} |\nabla u_i|^2 \mathrm{div} Y - \nabla_j u_i \nabla_l u_i Y^l_j \right) \\ &\quad + \int_{\mathbb{R}_+} \gamma(t) \int_\Omega \frac{F(u_i)}{\epsilon_i^2} \mathrm{div} Y.\end{aligned}$$

Therefore

$$\begin{aligned}\int_{\mathbb{R}_+} \gamma(t) \int_\Omega \delta V^i_t(Y) &= -\int_{\mathbb{R}_+} \gamma(t) \int_\Omega \langle Y \cdot \nabla u_i, \partial_t u_i \rangle \\ &\quad - \int_{\mathbb{R}_+} \gamma(t) \int_{\{x\in\Omega : |\nabla u_i|(x,t) \neq 0\}} 2 \left(\frac{\nabla_j u_i \nabla_l u_i Y^j_l}{|\nabla u_i|^2} \right) \frac{F(u_i)}{\epsilon_i^2}.\end{aligned}$$

By Lemma 8.21, we have

$$\lim_{i\to\infty}\int_{\Omega\times\mathbb{R}_+}|\gamma(t)|\,|\nabla Y|\,\frac{F(u_i)}{\epsilon_i^2}(x,t)=0.$$

Therefore, by taking i into infinity, we have

$$\begin{aligned}\int_{\mathbb{R}_+}\gamma(t)\delta V_t(Y) &= \lim_{i\to\infty}\int_{\mathbb{R}_+}\gamma(t)\int_\Omega Y\cdot\langle-\nabla u_i,\partial_t u_i\rangle\\ &= \int_{\mathbb{R}_+}\gamma(t)\int_\Omega\langle H_t,Y\rangle\,d\mu_t dt\end{aligned}$$

so that for L^1 a.e. $t\in\mathbb{R}_+$, $\delta V_t=H_t\mu_t$. □

For $V\in V_{n-2}^*(\Omega)$ and $x\in\Omega$, define

$$\Theta^{n-2}(\|V\|,x)\equiv\lim_{r\to 0}\frac{\|V\|(B_r(x))}{\alpha(n-2)r^{n-2}},\tag{9.17}$$

if the limit exists, where $\alpha(n-2)=|B_1^{n-2}|$ is the volume of the unit ball in $\mathbb{R}^{n-2}$.

We now derive a monotonicity formula for $V\in V_l^*(\Omega)$, with δV a Radon measure. The same formula was shown by Allard [2] Theorem 5.1 for classical l-varifolds $V\in V_l(\Omega)$.

Lemma 9.1.6 *Suppose $V\in V_l^*(\Omega)$ is such that $\|\delta V\|$ is a Radon measure on Ω. Then, for any $a\in supp(\|V\|)$ and $0<r<dist(a,\partial\Omega)$,*

$$\begin{aligned}\frac{d}{dr}\left(r^{-l}\|V\|(B_r(a))\right) &= r^{-l-2}\int_{\partial B_r(a)}|S^\perp(x)|^2\,dV(x,S)\\ &\quad -r^{-l-1}\lim_{\epsilon\downarrow 0}\delta V\left(\theta_\epsilon(|x|)x\right)\end{aligned}\tag{9.18}$$

where $\theta_\epsilon(|x|)\in C_0^1(B_r(a))$ converges to the characteristic function of $B_r(a)$ as $\epsilon\downarrow 0$, $|S^\perp(x)|^2=|x|^2-|S(x)|^2$, and $S(x):\mathbb{R}^n\to S$ is the orthogonal projection.

Proof. The proof is exactly as same as [2] Theorem 5.1. For $\theta_\epsilon(|x|)$ as above, one has

$$-\delta V(\theta_\epsilon(|x|)x)=\int_{B_r(a)\times A_{l,n}}\theta'(|x|)\left(1-\frac{|S^\perp(x)|^2}{|x|}\right)dV(x,S)+l\|V\|(\theta_\epsilon(|x|)).$$

This implies (9.18). □

As a consequence, we can show that $\Theta^{m-2}(\|V_t\|,\cdot)$ exists for L^1 a.e. $t\in\mathbb{R}_+$.

Corollary 9.1.7 *Suppose that $\{V_t\}_{t>0}$ is the family of generalized $(n-2)$ varifolds obtained as above. Then, for L^1 a.e. $t\in\mathbb{R}_+$, there exists a set $E_t\subset\Omega$, with $H^{n-2}(E_t)=0$, such that $\Theta^{n-2}(\|V_t\|,x)$ exists for any $x\in\Omega\setminus E_t$.*

Proof. First by Fatou's lemma and Lemma 9.1.5 we may assume that for L^1 a.e. $t \in \mathbb{R}_+$,

$$\lim_{i\to\infty} \int_\Omega |\partial_t u_i|^2 < \infty,$$

$$H_t \in L^2_{\|V_t\|}(\Omega, \mathbb{R}^n),\ \delta V_t = H_t \|V_t\|.$$

In particular,

$$\begin{aligned} \|\delta V_t\|\left(B_r(a)\right) &\leq \lim_{i\to\infty} \int_{B_r(a)} |\langle \partial_t u_i, \nabla u_i\rangle| \\ &\leq 2\left(\|V_t\|(B_r(a))\right)^{\frac{1}{2}} \lim_{i\to\infty}\left(\int_{B_r(a)} |\partial_t u_i|^2\right)^{\frac{1}{2}}. \end{aligned}$$

Hence Lemma 9.1.6 implies

$$\begin{aligned} &\frac{d}{dr}\left(r^{2-n}\|V_t\|\left(B_r(a)\right)\right) \\ &\geq r^{-n}\int_{\partial B_r(a)} |S^\perp(x)|^2\, dV_t(x,S) \\ &\quad -2\left(r^{2-n}\|V_t\|(B_r(a))\right)^{\frac{1}{2}}\left(\lim_{i\to\infty} r^{2-n}\int_{B_r(a)} |\partial_t u_i|^2\right)^{\frac{1}{2}} \\ &\geq r^{-n}\int_{\partial B_r(a)} |S^\perp(x)|^2\, dV_t(x,S) - r^{2-n}\|V_t\|\left(B_r(a)\right) \\ &\quad -r^{2-n}\lim_{i\to\infty}\int_{B_r(a)} |\partial_t u_i|^2. \end{aligned}$$

This implies

$$\begin{aligned} \frac{d}{dr}\left(e^r r^{2-n}\|V_t\|\left(B_r(a)\right)\right) &\geq r^{-n}\int_{\partial B_r(a)} \left|S^\perp(x)\right|^2 dV_t(x,S) \\ &\quad -r^{2-n}\lim_{i\to\infty}\int_{B_r(a)} |\partial_t u_i|^2. \end{aligned} \tag{9.19}$$

If we set

$$E_t = \left\{ a \in \Omega \mid \liminf_{r\to 0}\left(r^{2.5-n}\lim_{i\to\infty}\int_{B_r(a)} |\partial_t u_i|^2\right) \geq 1\right\}$$

then, for any $a \in \Omega \setminus E_t$, there exists $r_a > 0$ such that for any $0 < r \leq r_a$

$$r^{2-n}\lim_{i\to\infty}\int_{B_r(a)} |\partial_t u_i|^2 \leq 2r^{-\frac{1}{2}}.$$

Therefore, integrating (9.19) between $0 < r_1 \leq r_2 \leq r_a$ gives

$$\begin{aligned} &\left(e^{r_2} r_2^{2-n}\|V_t\|\left(B_{r_2}(a)\right) + \sqrt{r_2}\right) - \left(e^{r_1} r_1^{2-n}\|V_t\|\left(B_{r_1}(a)\right) + \sqrt{r_1}\right) \\ &\geq \int_{r_1}^{r_2} r^{-n}\int_{\partial B_r(a)} \left|S^\perp(x)\right|^2 dV_t(x,S). \end{aligned} \tag{9.20}$$

This implies that $\Theta^{n-2}(\|V_t\|, a)$ exists for all $a \in \Omega \setminus E_t$. By Vitali's covering lemma we have that $H^{n-2.5}(E_t) < \infty$ and hence $H^{n-2}(E_t) = 0$. This completes the proof. □

Note that by Theorem 7.2.4, Σ_t has locally finite H^{n-2}-measure for any $t > 0$. In fact we have

Lemma 9.1.8 *For L^1 a.e. $t \in \mathbb{R}_+$, there is $F_t \subset \Sigma_t$, with $H^{n-2}(F_t) = 0$, such that $\Theta^{n-2}(\|V_t\|, x) \geq \frac{\epsilon_0^2}{2}$ for all $x \in \Sigma_t \setminus F_t$, with $\epsilon_0 > 0$ given by Lemma 8.2.3.*

Proof. Define

$$G = \left\{ z \in \Sigma : \lim_{r \downarrow 0} \lim_{i \to \infty} \left(r^{2-n} \int_{P_r(z)} |\partial_t u_i|^2 \right) \geq \epsilon_0^5 \right\} \text{ and } G_t = G \cap \{t\}. \tag{9.21}$$

Then, by Vitali's covering lemma, we have $\mathcal{P}^{n-2}(G \cap P_R) < \infty$ for any $R > 0$. In particular, $\mathcal{P}^n(G) = 0$. Therefore for L^1 a.e $t \in \mathbb{R}_+$, $H^{n-2}(G_t) = 0$. Let $F_t = G_t \cup E_t$, where E_t is given by Lemma 9.1.7. Then we have $H^{n-2}(F_t) = 0$. For any $a \in \Sigma_t \setminus F_t$, there exists $r_a > 0$ such that

$$\lim_{i \to \infty} r^{2-n} \int_{P_r(a)} |\partial_t u_i|^2 < \epsilon_0^5, \ \forall 0 < r \leq r_a. \tag{9.22}$$

Since $a \in \Sigma_t$, it follows from Theorem 7.2.4 that there is a sufficiently large $K_0 > 0$ such that

$$\lim_{i \to \infty} r^{2-n} \int_{B_{2K_0 r}(a)} e(u_i)(x, t - r^2) \geq \frac{\epsilon_0^2}{2} \text{ for any } r \in (0, \frac{r_a}{2}].$$

On the other hand, Lemma 8.2.1 implies that for any $0 < r \leq \frac{r_a}{2K_0}$,

$$\begin{aligned}
\int_{B_{K_0 r}(a)} e(u_i)(x,t) \ &\geq \ \int_{B_{2K_0 r}(a)} e(u_i)(x, t - r^2) - \int_{P_{2K_0 r}(a)} |\partial_t u_i|^2 \\
&\quad - \left((K_0 r)^{-2} \int_{P_{2K_0 r}(a)} |\nabla u_i|^2 \right)^{\frac{1}{2}} \left(\int_{P_{2K_0 r}(a)} |\partial_t u_i|^2 \right)^{\frac{1}{2}} \\
&\geq \ \int_{B_{2K_0 r}(a)} e(u_i)(x, t - r^2) - C\epsilon_0^{2.5} r^{n-2} \geq \frac{\epsilon_0^2}{4} r^{n-2},
\end{aligned}$$

where we have used the fact

$$r^{-n} \int_{P_r(a)} |\nabla u_i|^2 \leq C, \quad \text{for all } 0 < r \leq r_a.$$

Thus we obtain that $\Theta^{n-2}(\|V_t\|, a) \geq \frac{\epsilon_0^2}{4}$. □

We are now ready to prove the slice rectifiability theorem.

Theorem 9.1.9 *Under the same notations as above, for L^1 a.e. $t \in \mathbb{R}_+$, $V_t \lfloor (\Sigma_t \times A_{n-2,n})$ is a $(n-2)$-rectifiable varifold, and Σ_t is a $(n-2)$-rectifiable set.*

Proof. Of course one can appeal the deep rectifiability theorem by Preiss [154] or modify the proof by Lin [123]. Here we give a generalized varifold approach similar to [2], which is conceptually simple.

First note that Lemma 9.1.8 implies that for L^1 a.e. $t \in \mathbb{R}_+$, there exists $F_t \subset \Sigma_t$, with $H^{n-2}(F_t) = 0$, such that $\frac{\epsilon_0^2}{4} \leq \Theta^{n-2}(\|V_t\|, x) < +\infty$ for any $x \in \Sigma_t \setminus G_t$. In fact,

$$F_t \subset \left\{ x \in \Sigma_t : \liminf_{r \downarrow 0} r^{2.5-n} \lim_{i \to \infty} \int_{B_r(x)} e(u_i) \geq 1 \right\}.$$

Hence $\hat{\Sigma}_t = \Sigma_t \setminus F_t$ can be decomposed as $\hat{\Sigma}_t = \cup_{j=1}^{\infty} \hat{\Sigma}_t^j$, where

$$\hat{\Sigma}_t^j = \bigcap_{0<r\leq \frac{1}{j}} \left\{ x \in \hat{\Sigma}_t : \lim_{i \to \infty} r^{2.5-n} \int_{B_r(x)} e(u_i) \leq 1 \right\}.$$

It suffices to show that for any $j \geq 1$, $\hat{\Sigma}_t^j$ is $(n-2)$-rectifiable. For this, we may assume that $H^{n-2}(\hat{\Sigma}_t^j) > 0$.

Applying Lemma 9.1.6 to $\hat{\Sigma}_t^j$, we can conclude that for all $a \in \hat{\Sigma}_t^j$,

$$\Theta^{n-2}(\|V_t\|, a, r) \equiv e^r r^{2-n} \|V_t\|(B_r(a)) + \sqrt{r} \tag{9.23}$$

is monotonically increasing for all $0 < r \leq \frac{1}{j}$. Therefore $\Theta^{n-2}(\|V_t\|, \cdot)$ is upper semi continuous on $\hat{\Sigma}_t^j$. In particular, $\Theta^{n-2}(\|V_t\|, y)$ is H^{n-2}-approximate continuous at H^{n-2} a.e. $x \in \hat{\Sigma}_t^j$ for $y \in \hat{\Sigma}_t^j$.

Note that if we represent $V_t = V_{x,t}\|V_t\|$, with $V_{x,t}$ a probability measure on $A_{n-2,n}$, then $V_{x,t}$ is H^{n-2}-measurable with value in the space of probability measures on $A_{n-2,n}$. Thus $V_{x,t}$ is H^{n-2}-approximate continuous for H^{n-2} a.e. $x \in \Sigma_t$ (cf. [55]). Therefore, for H^{n-2} a.e. $x_0 \in \hat{\Sigma}_t^j$, the following four properties hold:

$$\Theta^{*,n-2}\left(\hat{\Sigma}_t^j, x_0\right) = \limsup_{r \downarrow 0} r^{2-n} H^{n-2}(\Sigma_t \cap B_r(x_0)) \geq 2^{2-n} \tag{9.24}$$

$$\Theta^{n-2}(u(t), x_0) = \lim_{r \downarrow 0} r^{2-n} \int_{B_r(x_0)} |\nabla u(t)|^2 = 0, \tag{9.25}$$

$$\Theta^{n-2}(\|V_t\|, x) \text{ is } H^{n-2} - \text{approximate continuous at } x_0 \text{ for } x \in \hat{\Sigma}_t^j, \tag{9.26}$$

$$V_{x,t} \text{ is } H^{n-2} - \text{approximate continuous at } x_0 \text{ for } x \in \hat{\Sigma}_t^j, \tag{9.27}$$

$$\lim_{r \downarrow 0} \frac{\int_{B_r(x_0)} |H_t| \, d\|V_t\|}{r^{n-2}} = |H_t(x_0)| \, \Theta^{n-2}(\|V_t\|, x_0) < +\infty. \tag{9.28}$$

With (9.23), (9.24), (9.25) and (9.26) at hand, we can easily modify the proof of Lemma 4.2.6 without any difficulty to prove Lemma 8.5.1. More precisely, that there exist $s = s(n) \in (0, \frac{1}{2})$ and $0 < r_0 < \frac{1}{j}$ such that for any $0 < r \leq r_0$, there exist $(n-2)$-points $\{x_1, \cdots, x_{n-2}\} \subset \hat{\Sigma}_t^j \cap B_r(x_0)$ and $\epsilon(r) \downarrow 0$, as $r \downarrow 0$, such that
(i) $\Theta^{n-2}(\|V_t\|, x_j) \geq \Theta^{n-2}(V_t, x_j) - \epsilon(r)$ for $1 \leq j \leq n-2$.
(ii) $|x_1 - x_0| \geq sr$ and for $k \in \{2, \cdots, n-2\}$,

$$\text{dist}(x_k, x_0 + V_{k-1}) \geq sr, \; V_{k-1} = \text{span}\{x_1 - x_0, \cdots, x_{k-1} - x_0\}.$$

For $r_i \downarrow 0$ we can now show that after taking subsequence,

$$\mathcal{D}_{x_0,r_i}(V_t) \to V_{x_0,t}H^{n-2}\lfloor T \tag{9.29}$$

for some $(n-2)$-plane $T \subset \mathbb{R}^n$. Next we want to show that T is independent of the choice of $r_i \downarrow 0$. In fact, by (9.28) we can easily get

$$\lim_{i\to\infty} \|\delta(\mathcal{D}_{x_0,r_i}(V_{x,t}))\| = \lim_{r\to\infty} r_i^{3-n}(\mathcal{D}_{x_0,r_i})_\# \|\delta V_{x,t}\| = 0.$$

Therefore

$$\delta\left(V_{x_0,t}H^{n-2}\lfloor T\right) = 0$$

so that the constancy theorem for varifolds (see [189]) gives that $V_{x_0,t} = \delta_T$ and hence T is unique. Hence $\hat{\Sigma}_t^j$ is $(n-2)$-rectifiable for all $j \geq 1$. Therefore both Σ_t and $V_t\lfloor(\Sigma_t \times A_{n-2,n})$ are $(n-2)$-rectifiable. □

To conclude this section, we deduce some consequences of the above theorem to the steady states of heat flow of harmonic maps and Ginzburg-Landau heat flows. Let us first consider the Ginzburg-Landau equation, which can be viewed as the continuation of Example 9.1.4.

Corollary 9.1.10 *Under the same assumptions as in the Example 9.1.4, there exist a closed $(n-2)$-rectifiable set $\Sigma \subset \Omega$ and a H^{n-2}-measurable function $\epsilon_0^2 \leq \theta < \infty$ in Ω such that*
(i) $\nu(x) = \theta(x)H^{n-2}L\Sigma$ for H^{n-2} a.e. $x \in \Sigma$, and

$$V_{u_i} \to V \equiv V_u + V(\Sigma,\theta) \tag{9.30}$$

as convergences of on $V_{n-2}^*(\Omega)$, where

$$V(\Sigma,\theta) = \delta_{T_x\Sigma}\theta H^{n-2}\lfloor\Sigma.$$

Moreover, V is stationary in the sense that for any $Y \in C_0^1(\Omega,\mathbb{R}^n)$,

$$\int_\Omega \left(\frac{1}{2}|\nabla u|^2 div(Y) - \sum_{1\leq ij\leq n} u_i u_j Y_i^j\right) + \int_\Sigma div_\Sigma(Y)\theta\, dH^{n-2} = 0. \tag{9.31}$$

(ii) If, in additions, $N = S^{L-1}$, then

$$\theta(x) = \sum_{i=1}^{l_x} E(\phi_j, S^2) \text{ for } H^{n-2} \text{ a.e. } x \in \Sigma, \tag{9.32}$$

where $1 \leq l_x < \infty$ and $\phi_j : S^2 \to S^{L-1}$ are harmonic S^2's for $1 \leq j \leq l_x$.
(iii) Furthermore, if $N = S^2$ then $\theta(x) = 4\pi n_x$ for some positive integer n_x, for H^{n-2} a.e. $x \in \Sigma$. In particular, $\frac{1}{4\pi}V(\Sigma,\theta)$ is an integral $(n-2)$-varifold.

Proof. First note that by the static version of the discussion of §8.5 (see also Lin-Wang [134]) that the concentration set Σ of $\{u_i\}$ is given by

$$\Sigma = \left\{x \in \Omega \mid \epsilon_0^2 \le \Theta^{n-2}(\|V\|, x) < \infty\right\} = \left\{x \in \Omega \mid \Theta^{n-2}(\|V\|, x) > 0\right\}.$$

Moreover, as in Example 9.1.4 we have $\delta V = 0$ and theorem 9.1.9 then implies

$$V\lfloor\left\{(x, A) \mid \Theta^{n-2}(\|V\|, x) > 0,\ A \in A_{n-2,n}\right\} = \delta_{T_x\Sigma}\Theta^{n-2}(\|V\|, x) H^{n-2}\lfloor\Sigma$$

is a $(n-2)$-rectifiable varifold. In particular, Σ is a $(n-2)$-rectifiable set. Moreover, since $u_i \to u$ in $C^1_{\rm loc}(\Omega \setminus \Sigma, \mathbb{R}^L)$, we have

$$V\lfloor(\Omega \setminus \Sigma) \times A_{n-2,n} = \frac{1}{2}\delta_{A(u)}|\nabla u|^2(x)\, dx.$$

Therefore we obtain (9.31).
The conclusion of (ii) follows from Lin-Wang [135] Theorem B.
(iii) follows from (ii) and the fact that any nontrivial harmonic map from S^2 to S^2 has energy equal to $4\pi k$ for some positive integer k. □

Recall from §3 that if $u \in H^1(\Omega, N)$ a stationary harmonic map then

$$\int_\Omega \left(|\nabla u|^2 \mathrm{div}(X) - 2 \sum_{1 \le ij \le n} u_i u_j X^i_j \right) = 0,\ \forall X \in C^1_0(\Omega, \mathbb{R}^n). \tag{9.33}$$

By applying the quantization theorem by Lin-Riviere [131], we obtain

Corollary 9.1.11 *Let $\{u_i\} \subset H^1(\Omega, N)$ be stationary harmonic maps. Assume that $u_i \to u$ weakly in $H^1(\Omega, N)$, $\frac{1}{2}|\nabla u_i|^2(x)\, dx \to \frac{1}{2}|\nabla u|^2(x)\, dx + \nu$ for some nonnegative Radon measure ν on Ω, and $V_{u_i} \to V$ for some $V \in V^*_{n-2}(\Omega)$ on $V^*_{n-2}(\Omega)$. Then*
(i) there exist a $(n-2)$-rectifiable close set $\Sigma \subset \Omega$ and a H^{n-2} measurable function $\epsilon_0^2 \le \theta < \infty$ on Ω such that $\nu = \theta H^{n-2}\lfloor\Sigma$.
(ii) $V = V_u + V(\Sigma, \theta)$ is stationary in the sense that for any $Y \in C^1_0(\Omega, \mathbb{R}^n)$,

$$\int_\Omega \left(\frac{1}{2}|\nabla u|^2 \, div(Y) - \sum_{1 \le ij \le n} u_i u_j Y^j_i \right) + \int_\Sigma div_\Sigma(Y)\, \theta \, dH^{n-2} = 0. \tag{9.34}$$

(iii) If, in additions, $N = S^{L-1}$, then $\theta(x) = \sum_{i=1}^{l_x} E(\phi_j, S^2)$ for H^{n-2} a.e. $x \in \Sigma$, where $1 \le l_x < \infty$ and $\phi_j : S^2 \to S^{L-1}$ are bubbles. Furthermore, if $L = 3$ then $\theta(x) = 4\pi n_x$ for some positive integer n_x, for H^{n-2} a.e. $x \in \Sigma$. In particular, $\frac{1}{4\pi}V(\Sigma, \theta)$ is an integral $(n-2)$-varifold.

Remark 9.1.12 (9.34) was previously known for stationary solutions in the context of relaxed energies of harmonic maps by Brezis-Bethuel-Coron [14], and of Catersian currents of mappings into spheres by Giaquinta-Modica-Souček [69]. For stationary harmonic maps, (9.34) has previously been proved by Li-Tian [138] by a different method.

9.2 Generalized varifold flows and Brakke's motion

In this section, we will prove that the limiting pair $(u, \nu_t\, dt)$ satisfies a generalized varifold flow, to be defined below. The generalized varifold flow implies that $\{\nu_t\}_{t\geq 0}$ is a Brakke flow of $(n-2)$-rectifiable varifolds, if u is a *suitable* weak solution to the heat flow of harmonic maps (8.2) such that the equality of the energy inequality(8.74) holds. Note that a similar notion of suitable weak solutions of the Navies-Stokes equations in $\mathbb{R}^3$ was introduced by Cafferalli-Kohn-Nirenberg [21] (cf. also Lin [126]).

We first apply Theorem 9.1.9 to present V_t for L^1 a.e. $t \in \mathbb{R}_+$.

Lemma 9.2.1 *For L^1 a.e. $t \in \mathbb{R}_+$, we have*

$$V_t = \frac{1}{2}\delta_{A(u(\cdot,t))} \left|\nabla u(t)\right|^2 dx + V\left(\Sigma_t, \Theta^{n-2}(\|V_t\|, \cdot)\right). \tag{9.35}$$

Proof. It follows from the proof of Theorem 9.1.9 that for L^1 a.e. $t \in \mathbb{R}_+$, $\delta V_t = H_t\|V_t\|$, $H_t \in L^2_{\|V_t\|}(\Omega, \mathbb{R}^n)$, and $\epsilon_0^2 \leq \Theta^{n-2}(\|V_t\|, x) < \infty$ for H^{n-2} a.e. $x \in \Sigma_t$, $V_t\lfloor(\Sigma_t \times A_{n-2,n})$ is a $(n-2)$-varifold and

$$V_t\lfloor\Sigma_t = \delta_{T_x\Sigma_t}\Theta^{n-2}(\|V_t\|, x)H^{n-2}\lfloor\Sigma_t (\equiv V\left(\Sigma_t, \Theta^{n-2}\left(\|V_t\|, \cdot\right)\right).$$

Since $u_i \to u$ in $C^1_{\rm loc}(\Omega \setminus \Sigma_t, \mathbb{R}^L)$ and $V_{u_i(\cdot,t)} \to V_{u(\cdot,t)}$ on $\Omega \setminus \Sigma_t$, we have

$$V_t\lfloor(\Omega \setminus \Sigma_t) = \frac{1}{2}\delta_{A(u(\cdot,t))} \left|\nabla u(t)\right|^2 dx.$$

Combining these two together yields (9.35). □

The next lemma shows that $H_t(x) \in (T_x\Sigma_t)^\perp$.

Lemma 9.2.2 *For L^1 a.e. $t > 0$, we have*

$$H_t(x) \perp T_x\Sigma_t \quad \textit{for } H^{n-2} \textit{ a.e. } x \in \Sigma_t. \tag{9.36}$$

Proof. This can be proved by the Young measure method. Let M^{nL} denote the set of $n \times L$ matrices and consider Radon measures W_i on $\Omega \times \mathbb{R}_+ \times M^{nL}$ defined by

$$W_i(x, t, A) = \delta_{\frac{\nabla u_i}{|\nabla u_i|}(x,t)}(A)e(u_i)(x, t)\, dxdt.$$

Define $\phi : M^{nL} \to A_{n-2,n}$ by $\phi(A) = I_n - 2A^tA$. Then we have that $\phi_\#(W_i) = V_{u_i}$, where V_{u_i} is defined by (9.9). Since $V_{u_i} \to V = V_{x,t}\mu_t\, dt$ and $W_i \to W = W_{x,t}\mu_t\, dt$ for two probability measures $V_{x,t}$ on $A_{n-2,n}$ and $W_{x,t}$ on M^{nL}, $V_{x,t} = \phi_\#(W_{x,t})$. On the other hand, Lemma 9.2.1 implies that for L^1 a.e. $t \in \mathbb{R}_+$, $V_{x,t} = \delta_{T_x\Sigma_t}$ for $x \in \Sigma_t$. Hence we have that for H^{n-2} a.e. $x \in \Sigma_t$,

$$\int_{M^{nL}} (I_n - 2A^tA)\, dW_{x,t}(A) = \int_{A_{n-2,n}} A\, dV_{x,t}(A) = T_x\Sigma_t.$$

This implies that for any unit vector $e \in T_x\Sigma_t$,

$$\begin{aligned} 1 &= \langle e, e\rangle = \left\langle e, \int_{M^{nL}} (I_n - 2A^tA)(e)\, dW_{x,t}(A)\right\rangle \\ &= 1 - 2\int_{M^{nL}} |A(e)|^2\, dW_{x,t}(A). \end{aligned}$$

Hence for H^{n-2} a.e. $x \in \Sigma_t$, $|A(e)| = 0$ for $W_{x,t}$ a.e. $A \in M^{nL}$. Therefore for H^{n-2} a.e. $x \in \Sigma_t$,

$$\operatorname{supp}(W_{x,t}) \subset E(A) \equiv \left\{A = (A_1, \cdots, A_L)^t \mid \operatorname{span}\{A_1, \cdots, A_L\} \subset (T_x\Sigma_t)^\perp\right\}.$$

Note also that if we define $Z_i = \delta_{\frac{\nabla u_i}{|\nabla u_i|}} \langle \partial_t u_i, \nabla u_i\rangle\, dxdt$, then Z_i is absolutely continuous with respect to W_i. Assume that $Z_i \to Z$ on $\Omega \times \mathbb{R}_+ \times M^{nL}$. Then Z is absolutely continuous with respect to W and hence there exists a vector valued function $Z_{x,t}$ on M^{nL} such that $Z = Z_{x,t}W_{x,t}\mu_t\, dt$. Since

$$\left(\pi_{(x,t)}\right)_\# Z_i = \langle \partial_t u_i \nabla u_i\rangle\, dxdt \to -H_t(x)\mu_t\, dt,$$

we have

$$-H_t(x) = \int_{M^{nL}} Z_{x,t}(A)\, dW_{x,t}(A).$$

We now claim that for H^{n-2} a.e. $x \in \Sigma_t$, $Z_{x,t}(A) \in \operatorname{supp}(W_{x,t})$, which clearly implies $H_t(x) \in (T_x\Sigma_t)^\perp$. To see this, observe that $\langle \partial_t u_i, \nabla u_i\rangle \in E(\frac{\nabla u_i}{|\nabla u_i|})$ and hence

$$\int_{M^{nL}} \operatorname{dist}\left(A, \frac{dZ_i}{d\|Z_i\|}\right) d\|Z_i\|(A) = 0,$$

Taking i into ∞, this and the lower semicontinuity imply

$$\int_{M^{nL}} \operatorname{dist}\left(A, \frac{dZ}{d\|Z\|}\right) d\|Z\| = 0.$$

Hence for $W_{x,t}$ a.e. $A \in M^{nL}$, $Z_{x,t}(A) \in \operatorname{supp}(W_{x,t})$. □

Now we prove an energy inequality for the limiting map u and Radon measures ν, η.

Proposition 9.2.3 *Under the same notations as above, we have, for any $0 < t_1 < t_2 < \infty$ and $\phi \in C_0^1(\Omega, \mathbb{R}_+)$,*

$$\begin{aligned} &\left[\int_\Omega \frac{1}{2}\phi(x)|\nabla u|^2(x, t_2) + \nu_{t_2}(\phi)\right] - \left[\int_\Omega \frac{1}{2}\phi(x)|\nabla u|^2(x, t_1) + \nu_{t_1}(\phi)\right] \\ &\le -\int_{\Omega\times[t_1,t_2]} \left(|\partial_t u|^2\phi + \nabla\phi\langle\partial_t u, \nabla u\rangle\right) \\ &\quad - \int_{\Sigma_{t_1}^{t_2}} \left(\phi\, d\eta - \langle (T_x(\Sigma_t))^\perp \nabla\phi, H_t\rangle\, d\|V_t\|\, dt\right), \end{aligned} \tag{9.37}$$

where $\nu_t(\phi) = \int_\Omega \phi(x)\, d\nu_t(x)$, $\Sigma_{t_1}^{t_2} = \Sigma \cap (\mathbb{R}^n \times [t_1, t_2])$, and $(T_x\Sigma_t)^\perp$ denotes the normal space of Σ_t at x.

Proof. By taking i into infinity in (8.74) for u_i, we get

$$\left[\int_\Omega \frac{1}{2}\phi(x)|\nabla u|^2(x,t_2) + \nu_{t_2}(\phi)\right] - \left[\int_\Omega \frac{1}{2}\phi(x)|\nabla u|^2(x,t_1) + \nu_{t_1}(\phi)\right]$$
$$= -\int_{\Omega\times[t_1,t_2]} \phi|\partial_t u|^2 - \int_{\Sigma_{t_1}^{t_2}} \phi\, d\eta + \int_{\Omega\times[t_1,t_2]} \langle\nabla\phi, H_t\rangle\, d\|V_t\|\, dt. \tag{9.38}$$

Sine $\langle\partial_t u_i, \nabla u_i\rangle \to \langle\partial_t u, \nabla u\rangle$ strongly in $L^2_{\mathrm{loc}}(\Omega\times\mathbb{R}_+\setminus\Sigma)$, we have $H_t\, d\mu_t = -\langle\partial_t u, \nabla u\rangle\, dx$ on $\Omega\times\mathbb{R}_+\setminus\Sigma$. Therefore we have

$$\begin{aligned}\int_{\Omega\times[t_1,t_2]} \langle\nabla\phi, H_t\rangle\, d\|V_t\|\, dt &= -\int_{\Omega\times[t_1,t_2]} \langle\nabla\phi\partial_t u, \nabla u\rangle + \int_{\Sigma_{t_1}^{t_2}} \langle\nabla\phi, H_t\rangle\, d\mu_t \\ &= -\int_{\Omega\times[t_1,t_2]} \langle\nabla\phi\partial_t u, \nabla u\rangle \\ &\quad + \int_{\Sigma_{t_1}^{t_2}} \Big\langle (T_x\Sigma_t)^\perp\nabla\phi, H_t\Big\rangle\, d\mu_t.\end{aligned}$$

This gives (9.37). □

As a consequence, we have

Corollary 9.2.4 *Under the same notations as above, we have for any* $0 < t_1 < t_2 < \infty$ *and* $\phi \in C_0^1(\Omega, \mathbb{R}_+)$,

$$\left[\int_\Omega \frac{1}{2}\phi(x)|\nabla u|^2(x,t_2) + \nu_{t_2}(\phi)\right] - \left[\int_\Omega \frac{1}{2}\phi(x)|\nabla u|^2(x,t_1) - \nu_{t_1}(\phi)\right]$$
$$\le -\int_{\Omega\times[t_1,t_2]} \left(|\partial_t u|^2\phi + \langle\nabla\phi\partial_t u, \nabla u\rangle\right)$$
$$-\int_{\Sigma_{t_1}^{t_2}} \left(\frac{1}{2}|H_t|^2\phi - \langle (T_x(\Sigma_t))^\perp\nabla\phi, H_t\rangle\right) d\|V_t\|\, dt. \tag{9.39}$$

Proof. It suffices to prove

$$\int_{\Sigma_{t_1}^{t_2}} \phi\, d\eta \ge \frac{1}{2}\int_{\Sigma_{t_1}^{t_2}} |H_t|^2\phi\, d\nu_t\, dt. \tag{9.40}$$

To see it, note that for $\mathcal{P}^n$ a.e. $z_0 = (x_0, t_0) \in \Sigma$, by Cauchy-Schwarz inequality we have

$$\begin{aligned}|H_{t_0}(x_0)|^2 &\le \lim_{r\downarrow 0}\lim_{i\to\infty} \left(\frac{|\int_{P_r(z_0)}\langle\partial_t u_i, \nabla u_i\rangle|}{\int_{P_r(z_0)} e(u_i)}\right)^2 \\ &\le 2\lim_{r\downarrow 0}\lim_{i\downarrow\infty} \frac{\int_{P_r(z_0)}|\partial_t u_i|^2}{\int_{P_r(z_0)} e(u_i)}.\end{aligned}$$

Hence

$$\begin{aligned}|H_{t_0}(x_0)|^2\,d\mu(z_0) &\le 2\lim_{r\downarrow 0}\lim_{i\to\infty}\int_{P_r(z_0)}|\partial_t u_i|^2\\ &= 2\lim_{r\to 0}\left(\int_{P_r(z_0)}|\partial_t u|^2+\eta(P_r(z_0))\right)\le 2\,d\eta(z_0).\end{aligned}$$

This gives (9.39). □

We now introduce the notion of generalized varifold flow for a pair $(v, \eta_t\,dt)$.

Definition 9.2.5 For $v \in H^1_{\mathrm{loc}}(\Omega\times\mathbb{R}_+, N)\cap L^\infty(\mathbb{R}_+, H^1(\Omega, N))$ and a family of nonnegative Radon measures $\{\eta_t\}_{t\ge 0}$ in Ω, we say that $(v, \eta_t\,dt)$ is a generalized varifold flow, if the following holds:
(i) v is a weak solution of (8.2).
(ii) For L^1 a.e. $t\in\mathbb{R}_+$, $\eta_t = \|V_t\|$ for some $(n-2)$ rectifiable varifold $V_t\in V_{n-2}(\Omega)$, $\delta V_t = H_t\|V_t\|$, and $H_t \in L^2_{\|V_t\|}(\Omega,\mathbb{R}^n)$.
(iii) For any $0\le s\le t<\infty$ and $\phi\in C^1_0(\Omega,\mathbb{R}_+)$, we have

$$\begin{aligned}&\left[\int_\Omega \frac{1}{2}|\nabla v|^2(x,t)\phi(x)+\eta_t(\phi)\right]-\left[\int_\Omega \frac{1}{2}|\nabla v|^2(x,s)\phi(x)+\eta_s(\phi)\right]\\ &\le -\int_s^t\int_\Omega \left(|\partial_t v|^2\phi+\langle\nabla\phi\partial_t v,\nabla v\rangle\right)\\ &-\int_s^t\int_{A_t}\left(\phi(x)\,|H_t(x)|^2-\Big\langle (T_xA_t)^\perp\nabla\phi, H_t(x)\Big\rangle\right)d\eta_t\,dt,\end{aligned}\tag{9.41}$$

where $A_t = \mathrm{supp}(\eta_t)$.

One of the main theorems of this section is that the limiting pair $(u, \nu_t\,dt)$ is a generalized varifold flow.

Theorem 9.2.6 *Under the same assumption and notations and as above, the limiting pair $(u, \nu_t\,dt)$ is a generalized varifold flow.*

Proof. Comparing (9.39) and (9.2.5), it suffices to improve the $\frac{1}{2}$ factor in front $\int_s^t\int_{\Sigma_t}\phi(x)|H_t(x)|^2$ of (9.39) to 1. More precisely, we need to prove

Lemma 9.2.7 *Under the same assumptions and notations as above, we have*

$$\int_{\Sigma_t^s}\phi(x)\,|H_t(x)|^2\,d\nu_t(x)\,dt\le\int_{\Sigma_t^s}\phi(x)\,d\eta(x,t)\tag{9.42}$$

for any $0<t\le s<\infty$ and $\phi\in C^1_0(\Omega,\mathbb{R}_+)$.

Before proving (9.42), we would like to remark that (9.42) also follows from the energy quantization Theorem 9.3.1 below, which, however, is only proved for $N = S^{L-1}$ at the moment. Here we present a proof that is valid for any N.

Lemma 9.2.8 *For $\mathcal{P}^n$ a.e. $z = (x,t) \in \Sigma$, we have*

$$\lim_{r\downarrow 0}\lim_{i\to\infty} r^{-n}\int_{P_r(z)}\left(|\nabla_x u_i|^2 - |\nabla_y u_i|^2\right) = 0, \tag{9.43}$$

and

$$\lim_{r\downarrow 0}\lim_{i\to\infty} r^{-n}\int_{P_r(z)}\langle\nabla_x u_i\nabla_y u_i\rangle = 0, \tag{9.44}$$

where (x,y) denotes a coordinate of $(T_x\Sigma_t)^\perp \sim \mathbb{R}^2$.

Proof. Note first that from the proof of Theorem 9.3.1 below, we have that for $\mathcal{P}^n$ a.e. $z_0 = (x_0,t_0) \in \Sigma$,

$$\begin{aligned}
\lim_{r\downarrow 0}\lim_{i\to\infty} r^{-n}\int_{P_r(z_0)} e(u_i) &= \Theta^{n-2}(\|V_{t_0}\|, x_0) && (9.45)\\
\lim_{r\downarrow 0} r^{2-n}\int_{P_r(z_0)} |\partial_t u_i|^2 &= 0 && (9.46)\\
\lim_{r\downarrow 0} r^{-n}\int_{P_r(z_0)} \left(r^2|\partial_t u|^2 + |\nabla u|^2\right) &= 0 && (9.47)\\
\lim_{r\downarrow 0}\lim_{i\to\infty} r^{-n}\int_{P_r(z_0)} |\nabla_z u_i|^2 &= 0 && (9.48)
\end{aligned}$$

where $z \in T_{x_0}(\Sigma_{t_0}) \sim \mathbb{R}^{n-2}(\subset \mathbb{R}^n)$.
For such a $z_0 = (x_0,t_0)$, write $\mathbb{R}^n = \{X = (x,y,z) \mid (x,y) \in \mathbb{R}^2, z \in \mathbb{R}^{n-2}\}$. For $r_i \downarrow 0$, let $v_i(x,t) = u_i(x_0 + r_i x, t_0 + r_i^2 t) : P_2 \to \mathbb{R}^L$. Then v_i solves (9.12), with ϵ_i replaced by $\bar{\epsilon}_i = \frac{\epsilon_i}{r_i}(\downarrow 0)$, and

$$v_i(X,t) \to \text{constant in } C^1_{\text{loc}}(\mathbb{R}^{n+1}\setminus\mathbb{R}^{n-2}\times\mathbb{R},\mathbb{R}^L)$$

$$e(v_i)(X,t)\,dXdt \to \Theta^{n-2}(\|V_{t_0}\|,x_0)\,(H^{n-2}\lfloor\mathbb{R}^{n-2})\times(L^1\lfloor\mathbb{R}) \tag{9.49}$$

$$\left(|\nabla_x v_i|^2 - |\nabla_y v_i|^2\right)(X,t)\,dXdt \to \alpha(z,t)H^{n-2}\lfloor\mathbb{R}^{n-2}\times L^1\lfloor\mathbb{R}, \tag{9.50}$$

$$\langle\nabla_x v_i, \nabla_y v_i\rangle(X,t)\,dXdt \to \beta(z,t)H^{n-2}\lfloor\mathbb{R}^{n-2}\times L^1\lfloor\mathbb{R} \tag{9.51}$$

as convergence of Radon measures on P_2, for some measurable functions α and β on $\mathbb{R}^{n-2}\times\mathbb{R}$. Observe that (9.43) and (9.44) are equivalent to

$$\int_{B_1^{n-2}\times(-1,1)} \alpha(z,t)\,dzdt = \int_{B_1^{n-2}\times(-1,1)} \beta(z,t)\,dzdt = 0. \tag{9.52}$$

In order to prove (9.52), we recall the Pohozaev identity for v_i. For $X \in C^1_0(B^n_2,\mathbb{R}^n)$, multiplying the equation of v_i by $X\cdot\nabla v_i$ and integrating it by parts gives

$$\int_{P_2}\langle\partial_t v_i, \nabla v_i\cdot X\rangle = \int_{P_2}\left(e(v_i)\mathrm{div}(X) - \sum_{k,j}\langle\nabla_k v_i, \nabla_j v_i\rangle X_k^j\right). \tag{9.53}$$

By (9.46) we have

$$\lim_{i\to\infty}\int_{P_2}\langle\partial_t v_i, \nabla v_i\cdot X\rangle = 0$$

so that Lemma 8.21 implies

$$\int_{P_2} e(v_i)\mathrm{div}(X) = \int_{P_2} \frac{1}{2}|\nabla v_i|^2 \,\mathrm{div}(X) + O(1).$$

This, combined with (9.48), implies

$$\begin{aligned}
&\int_{P_2(0)} \frac{1}{2}\left(|v_{i,x}|^2 + |v_{i,y}|^2\right)\left(X^1_x + X^2_y\right) \\
&= \int_{P_2(0)} \left(|v_{i,x}|^2 X^1_x + |v_{i,y}|^2 X^2_y\right) + \langle v_{i,x}, v_{i,y}\rangle \left(X^2_x + X^1_y\right) + O(1). \quad (9.54)
\end{aligned}$$

In particular, we have

$$\int_{P_2(0)} \left(|v_{i,x}|^2 - |v_{i,y}|^2\right)\left(X^2_y - X^1_x\right) - 2\int_{P_2(0)} \langle v_{i,x}, v_{i,y}\rangle \left(X^2_x + X^1_y\right) = O(1). \quad (9.55)$$

Therefore we get

$$\int_{B_2^{n-2}\times(-4,4)} \left(\alpha(z,t)\left(X^2_y - X^1_x\right) - 2\beta(z,t)\left(X^2_x + X^1_y\right)\right) = 0 \quad (9.56)$$

for any X^1 and $X^2 \in C^1_0(B^n_2)$.

Now choosing $X^1(x,y,z) = x\phi(x,y,z)$ and $X^2 = 0$ for a suitable cut-off function $\phi \in C^1_0(B^n_2)$, we obtain

$$\int_{B_1^{n-2}\times(-1,1)} \alpha(z,t)\, dzdt = 0.$$

Similarly, choosing $X^2(x,y,z) = y\phi(x,y,z)$ and $X^1 = 0$ gives

$$\int_{B_1^{n-2}\times(-1,1)} \beta(z,t)\, dzdt = 0.$$

This proves (9.43) and (9.44). □

Proof of Lemma 9.42:
Note that Lemma 9.2.8 guarantees that for $\mathcal{P}^n$ a.e. $z_0 = (x_0, t_0) \in \Sigma$,

$$\int_{P_r(z_0)} |f_i|^2 = \int_{P_r(z_0)} |g_i|^2 = 1 + O(r, i^{-1}), \quad \int_{P_r(z_0)} \langle f_i, g_i\rangle = O(r, i^{-1}), \quad (9.57)$$

where

$$f_i = \frac{\sqrt{2}u_{i,x}}{\left(\int_{P_r(z_0)}(|u_{i,x}|^2 + |u_{i,y}|^2)\right)^{\frac{1}{2}}}, \quad g_i = \frac{\sqrt{2}u_{i,y}}{\left(\int_{P_r(z_0)}(|u_{i,x}|^2 + |u_{i,y}|^2)\right)^{\frac{1}{2}}}.$$

Therefore, applying the Parseval's inequality, we have

$$\lim_{r\downarrow 0}\lim_{i\to\infty} \int_{P_r(z_0)} |\partial_t u_i|^2 \geq \lim_{r\downarrow 0}\lim_{i\to\infty} \left[\left(\int_{P_r(z_0)} \langle \partial_t u_i, f_i\rangle\right)^2 + \left(\int_{P_r(z_0)} \langle \partial_t u_i, g_i\rangle\right)^2 \right].$$

Substituting f_i and g_i into this inequality and using the fact that

$$\lim_{r\downarrow 0}\lim_{i\to\infty} r^{-n}\int_{P_r(z_0)} |\nabla_z u_i|^2 = 0,$$

we have

$$\lim_{r\downarrow 0}\lim_{i\to\infty} \frac{(\int_{P_r(z_0)}\langle\partial_t u_i, \nabla u_i\rangle)^2}{\int_{P_r(z_0)}|\nabla u_i|^2} \le \frac{1}{2}\lim_{r\downarrow 0}\lim_{i\to\infty}\int_{P_r(z_0)} |\partial_t u_i|^2. \tag{9.58}$$

On the other hand, we have that for $\mathcal{P}^n$ a.e. $z_0 = (x_0, t_0) \in \Sigma$,

$$|H_{t_0}(x_0)|^2\, d\mu_t(z_0) \le \lim_{r\downarrow 0}\lim_{i\to\infty} 2\frac{|\int_{P_r(z_0)}\langle\partial_t u_i, \nabla u_i\rangle|^2}{\int_{P_r(z_0)}|\nabla u_i|^2}.$$

Therefore we have

$$\begin{aligned}|H_{t_0}(x_0)|^2\, d\mu(x_0, t_0) &\le \lim_{r\downarrow 0}\lim_{i\downarrow\infty}\int_{P_r(z_0)} |\partial_t u_i|^2\, dxdt \\ &= \lim_{r\downarrow 0}\int_{P_r(z_0)} |\partial_t u_i|^2 + \eta(P_r(z_0)) = \lim_{r\downarrow 0}\eta(P_r(z_0)).\end{aligned}$$

This clearly implies

$$\int_{\Sigma_t^s} \phi(x)|H_t(x)|^2\, d\mu_t(x)\, dt \le \int_{\Sigma_t^s}\phi(x)\, d\eta(x,t),\ \forall\phi\in C_0^1(\Omega,\mathbb{R}_+).$$

This completes the proof. □

We now introduce the notion of a *suitable* weak solution of the heat equation of harmonic maps.

Definition 9.2.9 A map $u \in H^1_{\rm loc}(\Omega\times\mathbb{R}_+, N)\cap L^\infty(\mathbb{R}_+, H^1(\Omega, N))$ is a suitable weak solution (8.2) if
(1) It is a weak solution of (8.2).
(2) It satisfies the energy equality: for any $0 \le t_1 < t_2 < \infty$ and $\phi\in C_0^1(\Omega,\mathbb{R}_+)$,

$$\begin{aligned}&\int_\Omega \frac{1}{2}|\nabla u|^2(x,t_2)\phi(x) - \int_\Omega\frac{1}{2}|\nabla u|^2(x,t_1)\phi(x) \\ &= -\int_{t_1}^{t_2}\int_\Omega \left(|\partial_t u|^2\phi + \langle\nabla\phi\cdot\partial_t u, \nabla u\rangle\right).\end{aligned} \tag{9.59}$$

It is easy to check that any smooth heat flow of harmonic map is a suitable weak solution to the heat flow of harmonic maps.

A direct consequence of Theorem 9.2.6 is

Corollary 9.2.10 *Under the same assumptions as in Theorem 9.2.6. If, in additions, the limiting map $u\in H^1_{\rm loc}(\Omega\times\mathbb{R}_+, N)$ is a suitable weak solution of (8.2), then the defect measure $\{\nu_t\}_{t\ge 0}$ satisfies: for any $0\le s\le t<\infty$ and $\phi\in C_0^1(\Omega, R_+)$,*

$$\nu_t(\phi) - \nu_s(\phi) \le -\int_s^t\int_{\Sigma_t}\left(\phi|H_t|^2 - \langle (T_x\Sigma_t)^\perp\nabla\phi, H_t\rangle\right) d\nu_t(x)\, dt. \tag{9.60}$$

Next we want to show that (9.60) implies that $\{\nu_t\}_{t\geq 0}$ is a Brakke flow. First, let us recall the definition of Brakke's flow by Ilmanen [99], which is slightly stronger than the one by Brakke [15].

Definition 9.2.11 For a Radon measure ν in Ω and $\phi \in C_0^2(\Omega, \mathbb{R}_+)$, set

$$\mathcal{B}(\nu, \phi) = -\int_\Omega \left(\phi|H|^2 - \langle (T_x\nu)^\perp \nabla\phi, H\rangle\right) d\nu$$

provided that the following three conditions hold:
(i) $\nu = \|V\|$ in $\{\phi > 0\}$ for some $V \in RV_{n-2}(\Omega)$, the set of $(n-2)$-rectifiable varifolds.
(ii) $\delta V = H\|V\|$ in $\{\phi > 0\}$.
(iii) $H \in L^2_{\|V\|}(\{\phi > 0\}, \mathbb{R}^n)$.
Otherwise, we set $\mathcal{B}(\nu, \phi) = -\infty$.

Definition 9.2.12 We say that a family of Radon measures $\{\mu_t\}_{t\geq 0}$ in Ω is a Brakke flow, if

$$\overline{\nabla}_t \mu_t(\phi) \equiv \limsup_{s\to t} \frac{\mu_s(\phi) - \mu_t(\phi)}{s-t} \leq \mathcal{B}(\mu_t, \phi) \tag{9.61}$$

for all $t \geq 0$ and $\phi \in C_0^2(\Omega, \mathbb{R}_+)$.

We now have our last theorem of this section.

Theorem 9.2.13 *Under the same assumption as Theorem 9.2.6, if, in additions, $u \in H^1_{\text{loc}}(\Omega \times \mathbb{R}_+, N)$ is a suitable weak solution to the heat flow of harmonic maps then $\{\nu_t\}_{t\geq 0}$ is a Brakke flow.*

Proof. First it follows from the previous section that for L^1 a.e. $t \in \mathbb{R}_+$, we have
(a) $\nu_t = \|V_t\|$ for some $V_t \in RV_{n-2}(\Omega)$,
(b) $\Theta^{n-2}(\|V_t\|, x) \geq \frac{\epsilon_0^2}{4}$ for H^{n-2} a.e. $x \in \Sigma_t$,
(c) $\delta V_t = H_t\|V_t\|$ with $H_t \in L^2_{\|V_t\|}(\Omega, R^m)$,
(d) $H_t(x) \perp T_x\|V_t\|$ for H^{n-2} a.e. $x \in \Sigma_t$.
Now we argue that (a)-(d) and (9.60) are sufficient to show (9.61) for $\{\nu_t\}_{t\geq 0}$. To see it, let us check the upper right derivative $\overline{\nabla}_+$ of ν_t for $t \geq 0$, the proof for lower right derivative is similar for $t > 0$. Let

$$L = \limsup_{s\downarrow t} \left(-\frac{1}{s-t}\int_t^s \int_\Omega \left(\phi|H_t|^2 - \langle\nabla\phi, H_t\rangle\right) d\nu_t\, dt\right).$$

Note that (9.60) implies $L \geq \overline{\nabla}_+\nu_t(\phi)$. If $L = -\infty$, then $\overline{\nabla}_+\nu_t(\phi) = -\infty$ so that (9.61) holds trivially. Hence we may assume that $L > -\infty$ and $\overline{\nabla}_+\nu_t(\phi) > -\infty$. Let $s_i \downarrow t$ be such that

$$\lim_{i\to\infty} \left(-\frac{1}{s_i - t}\int_t^{s_i} \int_\Omega \left(\phi|H_t|^2 - \langle\nabla\phi, H_t\rangle\right) d\nu_t\, dt\right) = L \tag{9.62}$$

and $t_i \in (t, s_i)$ be such that (a)-(d) hold at t_i, and

$$\int_\Omega \left(\phi|H_{t_i}|^2 - \langle (T_x\Sigma_{t_i})^\perp \nabla\phi, H_{t_i}\rangle\right) d\nu_{t_i} \le -L + O(1). \tag{9.63}$$

By the compactness theorem of Allard [2], we may assume that $V_{t_i} \to V$ in $\{\phi > 0\} \times G_{n-2,n}$ for some $V \in RV_{n-2}(\Omega)$. Moreover, by the result of Ilmanen [99] we have that $\|V\| = \nu_t$. There exists $H \in L^2_{\|V\|}(\Omega, \mathbb{R}^n)$ such that $\delta V = H\|V\| = H\nu_t$ and

$$\begin{aligned}&\int_\Omega \left(\phi|H|^2 - \langle T_x\Sigma_t, H\rangle\right) d\nu_t \\ &\le \liminf_{i\to\infty} \int_\Omega \left(\phi|H_{t_i}|^2 - \langle (T_x\Sigma_{t_i})^\perp \nabla\phi, H_{t_i}\rangle\right) d\nu_{t_i} = -L.\end{aligned}$$

Therefore

$$\overline{\nabla}_+\nu_t(\phi) \le L \le -\int_\Omega \left(\phi|H|^2 - \langle T_x\Sigma_t, H\rangle\right) d\nu_t = \mathcal{B}\left(\nu_t, \phi\right).$$

This completes the proof □

To conclude this section, we add another remark.

Remark 9.2.14 (1) It follows from Ambrosio-Soner [8], Proposition 5.3, that the Brakke flow is also a distance solution to the mean curvature flow. Therefore, under the condition that u is a suitable weak solution to the heat flow of harmonic maps, Theorem 9.2.6 implies that $\{\nu_t\}_{t\ge 0}$ is a distance solution to the mean curvature flow. (2) Under the assumption that u is a suitable weak solution to the heat flow of harmonic maps, if $\nu_0 = \alpha H^{n-2}\lfloor\Gamma_0$ for some $\alpha > 0$ and a closed $(n-2)$-dimensional Riemannian manifold Γ_0 and $\{\Gamma_t\}_{t\in[0,T)}$ is the smooth mean curvature flow, then there exists a non increasing function $\alpha : [0,T) \to [0,\alpha]$ such that $\nu_t = \alpha(t)H^{n-2}\lfloor\Gamma_t$ for $t \in [0,T)$.

9.3 Energy quantization of the defect measure

Throughout this section, we assume that $N = S^{L-1} \subset \mathbb{R}^L$ and $n \ge 3$. We will show that for $\mathcal{P}^n$ a.e. $z_0 = (x_0, t_0) \in \Sigma$, the density function $\Theta^{n-2}(\|V_{t_0}\|, x_0)$ is the finite sum of energies of harmonic S^2's. In the static case, this type of quantization result was first obtained by Lin-Riviereé [131] for stationary harmonic maps, and by Lin-Wang [135] for the static Ginzburg-Landau equations. Here we discuss the parabolic version of [131] and [135].

To better illustrate the analytic techniques, we consider the Ginzburg-Landau heat flows (the corresponding result for the heat flow of harmonic maps will be remarked in next section).

The main theorem of this section is

Theorem 9.3.1 *For $\mathcal{P}^n$ a.e. $z_0 = (x_0, t_0) \in \Sigma$,*

$$\Theta^{n-2}\left(\|V_{t_0}\|, x_0\right) = \sum_{i=1}^{l_{z_0}} E\left(\phi_i, S^2\right) \tag{9.64}$$

for some $1 \le l_{z_0} < \infty$, where $\phi_i : S^2 \to S^{L-1}$ $(1 \le i \le l_{z_0})$ are bubbles.

Proof. Let us first collect all the necessary facts we need, which can be found in §8.5 and [134]. More precisely, for L^1 a.e $t_0 \in \mathbb{R}_+$,

$$\lim_{i\to\infty} \int_\Omega |\partial_t u_i(t_0)|^2 \;<\; \infty, \tag{9.65}$$

$$\lim_{r\downarrow 0}\lim_{i\to\infty} r^{2-n} \int_{B_r(x)} |\partial_t u_i(t_0)|^2 \;<\; \infty, H^{n-2} \text{ a.e.} x \in \Sigma_{t_0}, \tag{9.66}$$

and for H^{n-2} a.e. $x_0 \in \Sigma_{t_0}$,

$$\epsilon_0^2 \le \Theta^{n-2}\left(\|V_{t_0}\|, x_0\right) < \infty, \tag{9.67}$$

$$\lim_{r\downarrow 0}\lim_{i\to\infty} \left(r^{2-n} \int_{P_r(z_0)} |\partial_t u_i|^2 \right) = 0, \tag{9.68}$$

$$\Theta^n(\mu, z) \text{ is } \mathcal{P}^n\text{-approximate continuous on at } z_0 = (x_0, t_0) \text{ for } z \in \Sigma_{t_0}, \tag{9.69}$$

$$\lim_{r\downarrow 0}\lim_{i\to\infty} r^{-n} \int_{P_r(z_0)} |\nabla_T u_i|^2 = 0, \;\; \forall T \in T_{x_0}\Sigma_{t_0}, \tag{9.70}$$

$$\lim_{r\downarrow 0} r^{-n} \int_{P_r(z_0)} \left(|\nabla u|^2 + r^2|\partial_t u|^2\right) = 0, \tag{9.71}$$

$$\Theta^{n-2}\left(\|V_{t_0}\|, x_0\right) \text{ is } H^{n-2}\text{-approximate continuous at } x_0. \tag{9.72}$$

Let $z_0 = (x_0, t_0) \in \Sigma$ be such that all of (9.65)–(9.72) hold. Assume $T_{x_0}\Sigma_{t_0} = \{(0,0)\} \times \mathbb{R}^{n-2} = \{(0,0,Y) : Y \in \mathbb{R}^{n-2}\}$ and write $x = (X, Y) \in \mathbb{R}^2 \times \mathbb{R}^{n-2}$ for $x \in \mathbb{R}^n$.
For any $r_i \downarrow 0$, define $v_i : P_2 \to \mathbb{R}^L$ by $v_i(x,t) = u_i(x_0 + r_i x, t_0 + r_i^2 t)$. Then we have

$$\lim_{n\to\infty} \int_{P_1(0)} \left(|\nabla_Y v_i|^2 + |\partial_t v_i|^2\right) = 0, \tag{9.73}$$

$$v_i \to \text{ constant weakly, but not strongly in } H^1(P_2, \mathbb{R}^L),$$

and

$$e(v_i)(X,Y,t)dX\,dY\,dt \to \overline{\nu}_t\,dt$$

as convergence of Radon measures on P_2.
As in §8.5 or [135], we have

Claim 1. $\overline{\nu}_t\,dt = \Theta^{n-2}(\|V_{t_0}\|, x_0) H^{n-2}\lfloor\mathbb{R}^{n-2}\times L^1\lfloor\mathbb{R}$ on P_1.
To see this, let $\phi\in C_0^1(B_1^2)$ and define $f_i, g_i, h_i : \mathbb{R}^{n-2}\times\mathbb{R}\to\mathbb{R}_+$ by

$$f_i(Y,t) = \int_{B_1^2} e(v_i)\,(X,Y,t)\,\phi(X)\,dX,\quad g_i(Y,t) = \int_{B_1^2} |\partial_t v_i|^2\,(X,Y,t)\;dX$$

and

$$h_i(Y,t) = \int_{B_1^2} |\nabla_Y v_i|^2\,(X,Y,t)\;dX.$$

Then (9.73) implies

$$\lim_{i\to\infty}\int_{B_1^{n-2}\times(-1,1)} (g_i+h_i)\,(Y,t)\;dY dt = 0. \tag{9.74}$$

For $1\le j\le n-2$, Y_j and t-derivative of f_i are

$$\begin{aligned}
&\frac{\partial f_i}{\partial Y_j}\\
=\;& \int_{B_1^2}\left(\langle\nabla_X v_i, \nabla^2_{XY_j} v_i\rangle - \frac{1}{\epsilon_i^2} f(v_i)\cdot\nabla_{Y_j} v_i\right)\phi + \int_{B_1^2}\left\langle\nabla_{Y_l} v_i, \nabla^2_{Y_lY_j} v_i\right\rangle\phi\\
=\;& -\int_{B_1^2}\phi\left(\Delta_X v_i + \frac{1}{\epsilon_i^2} f(v_i)\right)\nabla_{Y_j} v_i\\
&-\int_{B_1^2}\nabla_X\phi\cdot\left\langle\nabla_X v_i, \nabla_{Y_j} v_i\right\rangle + \int_{B_1^2}\left\langle\nabla_{Y_l} v_i, \nabla^2_{Y_lY_j} v_i\right\rangle\phi\\
=\;& -\int_{B_1^2}\phi\left\langle\partial_t v_i, \nabla_{Y_j} v_i\right\rangle - \int_{B_1^2}\nabla_X\phi\left\langle\nabla_X v_i, \nabla_{Y_j} v_i\right\rangle\\
&+\frac{\partial}{\partial Y_l}\int_{B_1^2}\phi\left\langle\nabla_{Y_l} v_i, \nabla_{Y_j} v_i\right\rangle = f_i^{1,j} + \mathrm{div}_{(Y,t)}(f_i^{2,j})
\end{aligned} \tag{9.75}$$

where

$$f_i^{1,j}(Y,t) = -\int_{B_1^2}\left(\nabla_X\phi\cdot\langle\nabla_X v_i, \nabla_{Y_j} v_i\rangle + \phi\langle\partial_t v_i, \nabla_{Y_j} v_i\rangle\right),$$

$$f_i^{2,j}(Y,t) = \left(\int_{B_1^2}\phi\langle\nabla_{Y_1} v_i, \nabla_{Y_j} v_i\rangle, \cdots, \int_{B_1^2}\phi\langle\nabla_{Y_{n-2}} v_i, \nabla_{Y_j} v_i\rangle, 0\right).$$

$$\begin{aligned}
\frac{\partial f_i}{\partial t} =\;& -\int_{B_1^2}\left\langle\Delta_X v_i + \frac{1}{\epsilon_i^2} f(v_i), \partial_t v_i\right\rangle\phi\\
&-\int_{B_1^2}\nabla_X\phi\langle\nabla_X v_i, \partial_t v_i\rangle + \int_{B_2^2}\phi\langle\nabla_{Y_i} v_i, \nabla_{Y_i}(\partial_t v_i)\rangle\\
=\;& -\int_{B_1^2}|\partial_t v_i|^2\phi - \int_{B_1^2}\langle\nabla_X v_i, \nabla_X\phi\cdot\partial_t v_i\rangle + \frac{\partial}{\partial Y_i}\int_{B_1^2}\phi\langle\nabla_{Y_i} v_i, \partial_t v_i\rangle\\
=\;& g_i^1 + \mathrm{div}_{(Y,t)}(g_i^2),
\end{aligned} \tag{9.76}$$

where

$$g_i^1(Y,t) = -\int_{B_1^2}\left(|\partial_t v_i|^2\,\phi + \langle\nabla_X v_i, \nabla_X\phi\cdot\partial_t v_i\rangle\right),$$

$$g_i^2(Y,t) = \left(\int_{B_1^2} \phi\langle \nabla_{Y_1} v_i, \partial_t v_i\rangle, \cdots, \int_{B_1^2} \phi\langle \nabla_{Y_{n-2}} v_i, \partial_t v_i\rangle, 0\right).$$

Note that (9.74) implies

$$\lim_{i\to\infty} \sum_{j=1}^{2} \left(\left\| f_i^j \right\|_{L^1(B_1^{n-2}\times(-1,1))} + \left\| g_i^j \right\|_{L^1(B_1^{n-2}\times(-1,1))} \right) = 0. \tag{9.77}$$

By (9.75), (9.76) and (9.77), we can apply the Allard's strong constancy Lemma 4.2.10 as in §4 and §8.5 to prove Claim 1. Moreover, we have

$$\lim_{i\to\infty} \left\| f_i(Y,t) - \Theta^{n-2}\left(\|V_{t_0}\|, x_0\right) \right\|_{L^1(B_1^{n-2}\times(-1,1))} = 0. \tag{9.78}$$

Therefore for any $\delta > 0$, there exists $E_\delta \subset B_1^{n-2} \times (-1,1)$, with $|E_\delta| \geq 1-\delta$, such that

$$\lim_{i\to\infty} \sup_{(Y,t)\in E_\delta} \left| f_i(Y,t) - \Theta^{n-2}\left(\|V_{t_0}\|, x_0\right) \right| = 0. \tag{9.79}$$

In order to prove that $\Theta^{n-2}(\|V_{t_0}\|, x_0)$ is the sum of energies of finitely many bubbles, it suffices to prove that $f_i(Y,t)$ converges to the sum of energies of finitely many bubbles for $(Y,t) \in E_\delta$.

We now define the local Hardy-Littlewood maximal function for f_i, h_i and p_i on $B_1^{n-2} \times (-1,1)$, where

$$p_i(Y,t) = \int_{B_1^2} \frac{1}{\bar{\epsilon_i}^2} F(v_i)\,(X,Y,t)\; dX,\;\; \bar{\epsilon}_i = \frac{\epsilon_i}{r_i} (\to 0).$$

Then by the weak L^1-estimate we have that there exists $F_\delta^i \subset B_1^{n-2} \times (-1,1)$, with $|F_\delta^i| \geq 1-\delta$, such that for any $(Y,t) \in F_\delta^i$,

$$\lim_{i\to\infty} M\left(g_i + h_i\right)(Y,t) = 0,\;\; \lim_{i\to\infty} M(f_i)(Y,t) \leq C\Theta^{n-2}\left(\|V_{t_0}\|, x_0\right), \tag{9.80}$$

and

$$\lim_{i\to\infty} M(p_i)(Y,t) = 0. \tag{9.81}$$

We also define the Hardy-Littlewood maximal function for

$$\tilde{g}_i(Y) = \int_{B_1^2\times[-1,1]} \left|\partial_t v_i\right|^2 (x,Y,t)\; dxdt, Y \in B_1^{n-2}.$$

Then there exists $G_\delta^i \subset B_1^{n-2}$, with $|G_\delta^i| \geq 1-\delta$, such that for any $Y \in G_\delta^i$,

$$\lim_{i\to\infty} M(\tilde{g}_i)(Y) = 0. \tag{9.82}$$

We need to prove that for any $(Y,t) \in E_\delta \cap F_\delta^i \cap (G_\delta^i \times [-1,1])$

$$\lim_{i\to\infty} f_i(Y,t) = \sum_{j=1}^{l} E(\phi_j, S^2) \tag{9.83}$$

for some $1 \leq l < \infty$, where $\phi_j : S^2 \to S^{L-1}$ $(1 \leq j \leq l)$ are bubbles.

Step 1. First Bubble
This step has been done in §8.5 (see also [134]). Here we briefly mention it. For any $(Y_i, t_i) \in E_\delta \cap F^i_\delta \cap (G^i_\delta \times [-1, 1])$, let $X_i \in B^2_{\frac{1}{2}}$ and $\delta_i > 0$ be such that

$$\begin{aligned} \int_{B^2_{\delta_i}(X_i)} e(v_i)(\cdot, Y_i, t_i) &= \frac{\epsilon_0^2}{C(n)} \\ &= \max\left\{ \int_{B^2_{\delta_i}(X)} e(v_i)(\cdot, Y_i, t_i) \mid X \in B^2_{\frac{1}{2}} \right\} \end{aligned} \tag{9.84}$$

where $\epsilon_0 > 0$ is given by Lemma 8.2.3 and $C(n) > 0$ is a large number.
As proved in §8.5, we have that $X_i \to 0$ and $\delta_i \to 0$. Moreover, by (9.79), (9.81), and (9.82), we have that for any $X \in B^2_{\frac{1}{2}}$,

$$(2\delta_i)^{-n} \int_{B^2_{2\delta_i}(X) \times B^{n-2}_{2\delta_n}(Y_i) \times (t_i - 4\delta_i^2, t_i + 4\delta_i^2)} e(v_i)(X, Y, t) \leq \epsilon_0^2, \tag{9.85}$$

and

$$\delta_i^{-n} \int_{B^2_{\delta_i}(X) \times B^{n-2}_{\delta_i}(Y_i) \times (t_i - \delta_i^2, t_i + \delta_i^2)} e(v_i)(X, Y, t) \geq \frac{\epsilon_0^2}{2}. \tag{9.86}$$

Set $w_i(X, Y, t) = v_i(X_i + \delta_i X, Y_i + \delta_i Y, t_i + \delta_i^2 t)$. Then Lemma (8.2.3) implies

$$w_i \to w \text{ in } C^1_{\text{loc}}\left(\mathbb{R}^2 \times B^{n-2}_2 \times (-4, 4), \mathbb{R}^L\right).$$

Moreover, $\partial_t w = \nabla_Y w = 0$, because of (9.79) and (9.82). Hence $w(X, Y, t) = w(X) : \mathbb{R}^2 \to S^{L-1}$ is a harmonic map with positive and finite energy, which can be lifted to a bubble ϕ_1. By repeating all the possible blowing-up at different points and scales, we can get

$$\Theta^{n-2}(\|V_{t_0}\|, x_0) = \lim_{i \to \infty} f_i(Y_i, t_i) \geq \sum_{j=1}^{l} E(\phi_j, S^2) \tag{9.87}$$

for some $l = l_{z_0} \leq \frac{\Theta^{n-2}(\|V_{t_0}\|, x_0)}{\epsilon_0^2}$, where $\phi_j : S^2 \to S^{L-1}$, $1 \leq j \leq l$, are bubbles.

Step 2. (9.87) is equality.
As in §6.4 and 6.5, it suffices to show that there is no energy concentration in the neck region between two bubbles at the same point. This approach here is motivated by [131] and [135]. The idea is to use the interpolation between $L^{2,1}$ and $L^{2,\infty}$ to control L^2 norm of ∇v_i in the neck region.
First, by an induction argument on l it suffices to show that (9.87) is an equality for $l = 1$.

Claim 2. For any $\epsilon > 0$ and sufficiently large $R > 0$, we have

$$\int_{B^2_{2r}(X_i) \setminus B^2_r(X_i)} e(v_i)(\cdot, Y_i, t_i) \leq \epsilon^2, \ \forall R\delta_i < r < \frac{1}{2}. \tag{9.88}$$

For, otherwise, one can do another rescaling to get a second bubble, which would contradict the assumption $l = 1$.
Applying Allard's strong constancy Lemma 4.2.10 with (9.79), (9.81) and (9.82) and Lemma 8.2.3, we obtain

$$e(v_i)\,(X,Y,t) \le \frac{C\epsilon^2}{|X-X_i|^2 + |Y-Y_i|^2 + |t-t_i|^2} \tag{9.89}$$

for $2R\delta_i \le |X - X_i| \le \frac{1}{4}$, $|Y - Y_i| \le \frac{|X-X_i|}{2}$, $|t-t_i| \le \frac{|X-X_i|^2}{4}$. In particular,

$$e(v_i)\,(X,Y,t) \le \frac{C\epsilon^2}{|X-X_i|^2} \tag{9.90}$$

for $2R\delta_i \le |X-X_i| \le \frac{1}{4}$, $|Y-Y_i| \le R\delta_i$, $|t-t_i| \le R^2\delta_i^2$. Hence by setting $w_i\,(X,Y,t) = v_i\left(X_i + \delta_i X, Y_i + \delta_i Y, t_i + \delta_i^2 t\right)$, we have

$$e(w_i)\,(X,Y,t) \le \frac{C\epsilon^2}{|X|^2},\ \forall 2R \le |X| \le \frac{1}{4\delta_i},\ \ |Y| \le R,\ |t| \le R^2. \tag{9.91}$$

This implies that $\nabla w_i(\cdot, Y, t) \in L^{2,\infty}(B^2_{(4\delta_i)^{-1}} \setminus B^2_{2R})$ for any $(Y,t) \in B_R^{n-2} \times (-R^2, R^2)$, and

$$\sup_{(Y,t)\in B_R^{n-2}\times(-R^2,R^2)} \|\nabla w_i(\cdot,Y,t)\|_{L^{2,\infty}(B^2_{(4\delta_i)^{-1}}\setminus B^2_{2R})} \le C\epsilon. \tag{9.92}$$

Now we estimate the $L^{2,1}$ norm of $\nabla\left(\frac{w_i}{|w_i|}\right)(\cdot,Y,t)$ over $B^2_{(4\delta_i)^{-1}}$.

Claim 3. For $\mathcal{P}^n$ a.e. $(Y,t) \in B_R^{n-2} \times (-R^2,R^2)$, $\nabla(\frac{w_i}{|w_i|})(\cdot,Y,t) \in L^{2,1}(B^2_{(4\delta_i)^{-1}})$. Moreover,

$$\begin{aligned}
&\int_{B_R^{n-2}\times(-R^2,R^2)} \left\|\nabla\left(\frac{w_i}{|w_i|}\right)(\cdot,Y,t)\right\|_{L^{2,1}(B^2_{(4\delta_i)^{-1}})} dY\,dt \\
&\le C\delta_i^{-n} \int_{B^2_{\frac{1}{2}}(X_i)\times B^{n-2}_{R\delta_i}(Y_i)\times(t_i - R^2\delta_i^2, t_i + R^2\delta_i^2)} \left(|\nabla v_i|^2 + |\partial_t v_i|^2\right) \le C.
\end{aligned} \tag{9.93}$$

Proof. It is similar to the proof of [135] theorem B. Here we only sketch it. For any $t \in (-R^2, R^2)$, denote $\omega_i(X,Y) = w_i(X,Y,t) : B^2_{(2\delta_i)^{-1}} \times B^{n-2}_{2R} \to \mathbb{R}^L$, then we have

$$\Delta\omega_i + \frac{1}{\epsilon_i^2}\left(1 - |\omega_i|^2\right)\omega_i = l_i,\ \ l_i(X,Y) = \partial_t w_i\,(X,Y,t)\,.$$

For $1 \le j, l \le L$, let α_i^{jl} be the 1-form defined by $\alpha_i^{jl} = d\omega_i^j \omega_i^l - \omega_i^j d\omega_i^l$. Then

$$d^*\alpha_i^{jl} = \Delta\omega_i^j\omega_i^l - \Delta\omega_i^l\omega_i^j = l_i^j\omega_i^l - l_i^l\omega_i^j \equiv H_i^{jl},$$

$$\Delta\alpha_i^{jl} = dH_i^{jl} + 2d^*\left(d\omega_i^j \wedge d\omega_i^l\right). \tag{9.94}$$

Let $\tilde{\omega}_i : \mathbb{R}^n \to \mathbb{R}^L$ be an extension of ω_i such that

$$\|\nabla\tilde{\omega}_i\|_{L^2(\mathbb{R}^n)} \le C\|\nabla\omega_i\|_{L^2(B^2_{(2\delta_i)^{-1}}\times B^{n-2}_{2R})} \tag{9.95}$$

and $\overline{H}_i^{jl} : \mathbb{R}^n \to \mathbb{R}$ be an extension of H_i^{jl} such that $\overline{H}_i^{jl} = 0$ outside $B^2_{(2\delta_i)^{-1}} \times B^{n-2}_{2R}$. Let $F_i^{jl} \in H^1(\mathbb{R}^n, \wedge^2(\mathbb{R}^n))$ solve

$$\Delta F_i^{jl} = 2d\tilde{\omega}_i^j \wedge d\tilde{\omega}_i^l \ \text{ in } \mathbb{R}^n. \tag{9.96}$$

Then by Theorems 3.2.5 and 3.2.4, we have that $F_i^{jl} \in W^{2,1}(\mathbb{R}^n, \wedge^2(\mathbb{R}^n))$ and

$$\begin{aligned}\left\|\nabla^2 F_i^{jl}\right\|_{L^1(\mathbb{R}^n)} &\le C\left\|d\tilde{\omega}_i^j \wedge d\tilde{\omega}_i^l\right\|_{\mathcal{H}^1(\mathbb{R}^n)} \le C\left\|\nabla\tilde{\omega}_i\right\|^2_{L^2(\mathbb{R}^n)} \\ &\le C\|D\omega_i\|^2_{L^2(B^2_{(2\delta_i)^{-1}}\times B^{n-2}_{2R})}.\end{aligned} \tag{9.97}$$

Let $G_i^{jl} \in H^1(\mathbb{R}^n)$ solve

$$\Delta G_i^{jl} = \overline{H}_i^{jl} \ \text{ in } \mathbb{R}^n. \tag{9.98}$$

Then we have that $\nabla^2 G_i^{jl} \in L^2(\mathbb{R}^n)$ and

$$\left\|\nabla^2 G_i^{jl}\right\|_{L^2(\mathbb{R}^n)} \le C\left\|\overline{H}_i^{jl}\right\|_{L^2(\mathbb{R}^n)} \le C\left\|\partial_t w_i\right\|_{L^2(B^2_{(2\delta_i)^{-1}}\times B^{n-2}_{2R})}. \tag{9.99}$$

In particular, by the Hölder inequality we have

$$\begin{aligned}\left\|\nabla^2 G_i^{jl}\right\|_{L^1(B^2_{(2\delta_i)^{-1}}\times B^{n-2}_{2R})} &\le C\left\|\nabla^2 G_i^{jl}\right\|_{L^2(B^2_{(2\delta_i)^{-1}}\times B^{n-2}_{2R})}(R^{n-2}\delta_i^{-2})^{\frac12} \\ &\le C(R^{n-2}\delta_i^{-2})^{\frac12}\left\|\partial_t w_i\right\|_{L^2(B^2_{(2\delta_i)^{-1}}\times B^{n-2}_{2R})}.\end{aligned} \tag{9.100}$$

Since

$$\alpha_i^{jl} = dG_i^{jl} + 2d^*(F_i^{jl}) + K_i^{jl}$$

where K_i^{jl} is a harmonic 1-form with $i^*(\alpha_i^{jl} - 2d^*F_i^{jl} - dG_i^{jl}) = 0$, and $i : \partial(B^2_{(2\delta_i)^{-1}} \times B^{n-2}_{\frac{3R}{2}}) \to \mathbb{R}^n$ denotes the inclusion map. By suitably choosing $R > 0$ and using Fubini's theorem, we may assume that

$$\|\alpha_i^{jl}\|_{L^1(\partial(B^2_{(2\delta_i)^{-1}}\times B^{n-2}_{\frac{3R}{2}}))} \le CR^{-1}\|\nabla\omega_i\|_{L^1(B^2_{\delta_i^{-1}}\times B^{n-2}_{2R})},$$

$$\begin{aligned}&\left\|(|\nabla^2 G_i^{jl}| + |\nabla^2 F_i^{jl}|)\right\|_{L^1(\partial(B^2_{(2\delta_i)^{-1}}\times B^{n-2}_{\frac{3R}{2}}))} \\ &\le CR^{-1}\|(|\nabla^2 G_i^{jl}| + |\nabla^2 F_i^{jl}|)\|_{L^1((B^2_{\delta_i^{-1}}\times B^{n-2}_{2R}))}.\end{aligned}$$

Therefore, by the standard estimate on harmonic forms, we have

$$\|\nabla K_i^{jl}\|_{L^1(B^2_{(4\delta_i)^{-1}}\times B^{n-2}_R)} \le C\left(R^{n-2}\delta_i^{-2}\right)^{\frac12}\left(\int_{B^2_{\delta_i^{-1}}\times B^{n-2}_R}(|\nabla w_i|^2 + |\partial_t w_i|^2)\right)^{\frac12}.$$

Hence for H^{n-2} a.e. $Y \in B_R^{n-2}$, by $W^{1,1}(\mathbb{R}^2) \subset L^{2,1}(\mathbb{R}^2)$, we have $\alpha_i^{jl}(\cdot, Y) \in L^{2,1}(B^2_{(4\delta_i)^{-1}})$ and

$$\begin{aligned}\left\|\alpha_i^{jl}(\cdot,Y)\right\|_{L^{2,1}(B^2_{(4\delta_i)^{-1}})} &\le C\left\|\nabla\alpha_i^{jl}\right\|_{W^{1,1}(B^2_{(4\delta_i)^{-1}})}\\ &\le C\left\|(|\nabla^2 G_i^{jl}| + |\nabla^2 F_i^{jl}| + |\nabla K_i^{jl}|)\right\|_{L^1(B^2_{(4\delta_i)^{-1}})}\end{aligned}$$

so that

$$\begin{aligned}&\int_{B_R^{n-2}} \left\|\alpha_i^{jl}(\cdot,Y)\right\|_{L^{2,1}(B^2_{(4\delta_i)^{-1}})} dY\\ &\le CR^{n-2}\delta_i^{-2}\int_{B^2_{\delta_i^{-1}}\times B^{n-2}_{2R}} \left(|\nabla w_i|^2 + |\partial_t w_i|^2\right)(X,Y,t)\,dXdY. \end{aligned} \tag{9.101}$$

Hence by the duality between $L^{2,1}$ and $L^{2,\infty}$ we obtain

$$\begin{aligned}&\int_{(B^2_{(4\delta_i)^{-1}}\setminus B^2_{2R})\times B_R^{n-2}} \left|\alpha_i^{jl}\right|^2(X,Y)\,dXdY\\ &\le \int_{B_R^{n-2}} \left\|\alpha_i^{jl}(\cdot,Y)\right\|_{L^{2,1}(B^2_{(4\delta_i)^{-1}})} \left\|\alpha_i^{jl}(\cdot,Y)\right\|_{L^{2,\infty}(B^2_{(4\delta_i)^{-1}}\setminus B^2_{2R})} dY\\ &\le \sup_{Y\in B_R^{n-2}} \left\|\alpha_i^{jl}(\cdot,Y)\right\|_{L^{2,\infty}(B^2_{(4\delta_i)^{-1}}\setminus B^2_{2R})} \int_{B_R^{n-2}} \left\|\alpha_i^{jl}(\cdot,Y)\right\|_{L^{2,1}(B^2_{(4\delta_i)^{-1}})} dY\\ &\le C\epsilon R^{n-2}\delta_i^{-2}\int_{B^2_{\delta_i^{-1}}\times B_R^{n-2}} \left(|\nabla w_i|^2 + |\partial_t w_i|^2\right)(X,Y,t)\,dXdY. \end{aligned} \tag{9.102}$$

Observe that

$$\sum_{jl}\left|d\omega_i^j\omega_i^l - \omega_i^j d\omega_i^l\right|^2 = |\omega_i|^2\left|\nabla(\frac{\omega_i}{|\omega_i|})\right|^2 \quad \text{and } |\omega_i| \ge \frac{1}{2} \text{ on } B^2_{\delta_i^{-1}}\times B^{n-2}_{2R}.$$

Thus, integrating over $t \in (-R^2, R^2)$, (9.102) implies

$$\begin{aligned}&\int_{(B^2_{(4\delta_i)^{-1}}\setminus B^2_{2R})\times B_R^{n-2}\times(-R^2,R^2)} \left|\nabla(\frac{w_i}{|w_i|})\right|^2(X,Y,t)\\ &\le C\epsilon R^{n-2}\delta_i^{-2}\int_{B^2_{\delta_i^{-1}}\times B_R^{n-2}\times(-R^2,R^2)} \left(|\nabla w_i|^2 + |\partial_t w_i|^2\right). \end{aligned} \tag{9.103}$$

Finally, we need to control the L^2 norm of $\nabla|w_i|$. Since $|w_i| \ge \frac{1}{2}$, we can write $w_i = \rho_i\theta_i$, with $\rho_i \ge \frac{1}{2}$ and $\theta_i \in S^{L-1}$. Then

$$\Delta\rho_i + \frac{1}{\epsilon_i^2}\left(1-\rho_i^2\right)\rho_i - \rho_i|\nabla\theta_i|^2 = \langle\partial_t w_i, \theta_i\rangle. \tag{9.104}$$

Multiplying (9.104) by $(1-\rho_i)$ and integrating it over $(B^2_{(4\delta_i)^{-1}}\setminus B^2_{2R})\times B^{n-2}_R\times(-R^2,R^2)$, we obtain

$$\begin{aligned}
&\int_{(B^2_{(4\delta_i)^{-1}}\setminus B^2_{2R})\times B^{n-2}_R\times(-R^2,R^2)} |\nabla\rho_i|^2\\
\leq\ & C\epsilon_i^{-2}\int_{B^2_{\delta_i^{-1}}\times B^{n-2}_R\times(-R^2,R^2)} (1-\rho_i^2)^2\\
&+C\int_{(B^2_{(4\delta_i)^{-1}}\setminus B^2_{2R})\times B^{n-2}_R\times(-R^2,R^2)}\left(|\nabla(\frac{w_i}{|w_i|})|^2+|\partial_t w_i|^2\right)+\text{boundary terms}\\
\leq\ & C\epsilon+O(1),
\end{aligned}$$

where we have used Fubini's theorem and above estimates to show the boundary term converges to zero. In particular, we get

$$R^{-n}\int_{(B^2_{(4\delta_i)^{-1}}\setminus B^2_{2R})\times B^{n-2}_R\times(-R^2,R^2)} |\nabla w_i|^2(X,Y,t)\leq C\epsilon. \tag{9.105}$$

This, combines with the Allard's strong constancy Lemma 4.2.10, implies

$$\int_{B^2_{(4\delta_i)^{-1}}(X_i)\setminus B^2_{2R}(X_i)} |\nabla w_i|^2(X,Y_i,t_i)\leq C\epsilon. \tag{9.106}$$

This finishes Step 2 and hence Theorem 9.3.1 is proven. □

Next we would like to discuss the quantization at $t=+\infty$. Assume that $t_i\uparrow\infty$ is such that

$$\lim_{i\uparrow\infty}\left(\int_{t_i-1}^{t_i+1}\int_\Omega|\partial_t u_i|^2+\int_\Omega|\partial_t u_i|^2(x,t_i)\right)=0, \tag{9.107}$$

$$\lim_{i\uparrow\infty}\int_\Omega\frac{1}{\epsilon_i^2}F(u_i)(x,t_i)=0. \tag{9.108}$$

Assume also that $u_i(t_i)\to u_\infty$ weakly in $H^1(\Omega,\mathbb{R}^L)$ and

$$e(u_i)(x,t_n)\,dx\to\mu_\infty\equiv\frac{1}{2}|\nabla u_\infty|^2(x)\,dx+\nu_\infty$$

as convergence of Radon measures on Ω for some nonnegative Radon measure ν_∞ in Ω. Moreover, $V_{u_i(t_i)}\to V_\infty$ in $V^*_{n-2}(\Omega)$, $\|V_\infty\|=\mu_\infty$. It follows from Example 9.1.3 that $\delta V_\infty=0$. Therefore by Lemma 9.1.6 we have that for all $a\in\operatorname{supp}(\|V_\infty\|)$ and $0<r\leq R<\operatorname{dist}(a,\partial\Omega)$,

$$\begin{aligned}
&R^{2-n}\|V_\infty\|(B_R(a))-r^{2-n}\|V_\infty\|(B_r(a))\\
\geq\ &\int_{B_R(a)\setminus B_r(a)}|y-a|^{-n-2}\left|S^\perp(y)\right|^2 dV(y,S).
\end{aligned}\tag{9.109}$$

In particular, $\Theta^{n-2}(\|V_\infty\|,x)$ exists for all $x\in\operatorname{supp}(\|V_\infty\|)$. Now define

$$\Sigma^1_\infty=\left\{x\in\Omega\mid\Theta^{n-2}(\|V_\infty\|,x)\geq\epsilon_1^2\right\}$$

$$\Sigma^2_\infty = \left\{ x \in \Omega \mid \lim_{r\downarrow 0} \lim_{i\uparrow\infty} r^{2-n} \int_{P_r(x,t_i)} |\partial_t u_i|^2 > 0 \right\}.$$

Then we have

$$H^{n-2}(\Sigma^1_\infty) < \infty \quad\text{and}\quad H^{n-2}(\Sigma^2_\infty) = 0.$$

We need an ϵ_0-regularity estimate at $t = t_i$.

Claim 4. There exist $\epsilon_1 > 0$ and $\delta_1 > 0$ such that for any $x \in \Omega$, if

$$r^{2-n} \int_{B_r(x)} e(u_i)(x,t_i)\,dx \le \epsilon_1^2,\ r^{2-n} \int_{P_r(x,t_i)} |\partial_t u_i|^2 \le \epsilon_1^4. \tag{9.110}$$

Then

$$(\delta_1 r)^2 \sup_{B_{\delta_1 r}(x)} e(u_i)(x,t_i) \le C\epsilon_1^2. \tag{9.111}$$

Proof. It follows from Lemma 7.5.2 that for any $t_i - r^2 \le t \le t_i$

$$\begin{aligned} & r^{2-n} \int_{B_r(x)} e(u_i)(x,t_i) - r^{2-n} \int_{B_{\frac{r}{2}}(x)} e(u_i)(x,t) \\ \ge\ & -r^{2-n} \int_{P_r(x,t_i)} |\partial_t u_i|^2 - C\left(r^{-n} \int_{P_r(x,t_i)} |\nabla u_i|^2 \right)^{\frac{1}{2}} \\ & \cdot \left(r^{2-n} \int_{P_r(x,t_i)} |\partial_t u_i|^2 \right)^{\frac{1}{2}} \\ \ge\ & -C\epsilon_1^2. \end{aligned}$$

Therefore we have for all $t \in [t_i - r^2, t_i]$

$$(\frac{r}{2})^{2-n} \int_{B_{\frac{r}{2}}(x)} e(u_i)(x,t) \le C\epsilon_1^2,$$

and

$$(\frac{r}{2})^{-n} \int_{P_{\frac{r}{2}}(x,t_i)} e(u_i) \le C\epsilon_1^2.$$

Therefore, by choosing ϵ_1 sufficiently small and applying Lemma 8.2.3, the claim follows.

Claim 5. $\Sigma_\infty = \Sigma^1_\infty \cup \Sigma^2_\infty$ is closed and has finite H^{n-2} measure, and $u_i \to u_\infty$ in $C^1_{\rm loc}(\Omega \setminus \Sigma_\infty, \mathbb{R}^L)$.

Proof. For any $x_0 \in \Omega \setminus \Sigma_\infty$, there exist $r_0 > 0$ and $i_0 >> 1$ such that for all $i \ge i_0$

$$r_0^{2-n} \int_{B_{r_0}(x_0)} e(u_i)(x,t_i) \le r_0^{2-n} \|V_\infty\|(B_{r_0}(x_0)) + \epsilon_1^2 \le 2\epsilon_1^2,$$

and

$$r_0^{2-n} \int_{P_{r_0}(x_0,t_i)} |\partial_t u_i|^2 \le \epsilon_1^4.$$

Therefore by Claim 4 we have that for $i \geq i_0$,

$$\sup_{B_{\delta_1 r_0}(x_0)} e(u_i)(x, t_i) \leq C\epsilon_1^2$$

so that $B_{\delta_1 r_0}(x_0) \cap \Sigma_\infty = \emptyset$. Hence Σ_∞ is closed and $u_i \to u_\infty$ in $C^1_{\rm loc}(\Omega \setminus \Sigma, \mathbb{R}^L)$. This and (9.107) imply that u_∞ is a weakly harmonic map with $\text{sing}(u_\infty) \subset \Sigma_\infty$.

We now have

Theorem 9.3.2 *Under the same notations as above, we have*
(i) Σ_∞ is a closed $(n-2)$-rectifiable set.
(ii) If, in additions, $N = S^{L-1}$, then for H^{n-2} a.e. $x \in \Sigma_\infty$, there exist $1 \leq l_x \leq \frac{E_0}{\epsilon_0^2}$ and l_x-many bubbles $\{\phi_j\}_{j=1}^{l_x}$ such that

$$\Theta^{n-2}(\|V_\infty\|, x) = \sum_{j=1}^{l_x} E(\phi_j, S^2). \tag{9.112}$$

Proof. (i) follows from the fact that V_∞ is stationary and Theorem 9.1.9. (ii) is similar to that of Theorem 9.3.1. The only change that we need to make is to show: for H^{n-2} a.e. $x_0 \in \Sigma_\infty$,

$$\lim_{r \downarrow 0} \lim_{i \uparrow \infty} r^{2-n} \int_{B_r(x_0)} |D_T u_i|^2 = 0 \quad \text{for all } T \in T_{x_0}\Sigma_\infty. \tag{9.113}$$

But (9.113) follows from Lemma 8.5.1. Then we can follow lines by lines of the proof of Theorem 9.3.1 to show (9.112). □

9.4 Further remarks

In this section, we consider the class $\mathcal{A}$ of suitable weak solutions to the heat flow of harmonic maps. The goal of this section is to remark that all the results from §9.1 to §9.3 remain to be true in $\mathcal{A}$ with proofs almost as same as those in §9.1-§9.3. Suppose $\{u_i\} \subset \mathcal{A}$ satisfies the bound:

$$\sup_{0<t<\infty} \left(\int_0^t \int_\Omega |\partial_t u_i|^2 + E(u_i(\cdot, t)) \right) \leq E(u_0) \quad \text{for all } i. \tag{9.114}$$

Then we may assume that $u_i \to u$ weakly in $H^1_{\rm loc}(\Omega \times \mathbb{R}_+, N)$ and

$$\frac{1}{2} |\nabla u_i|^2 (x, t)\, dxdt \to \frac{1}{2} |\nabla u|^2 (x, t)\, dxdt + \nu (\equiv \mu),$$

$$|\partial_t u_i|^2 (x, t)\, dxdt \to |\partial_t u|^2 (x, t)\, dxdt + \eta$$

for two nonnegative Radon measures $\nu = \nu_t\, dt$ and η on $\Omega \times \mathbb{R}_+$. Let Σ be the concentration set defined as in §8.2, with $e(u_i)$ replaced by $\frac{1}{2}|\nabla u_i|^2$.
As in §8.3, for $z_0 \in \Sigma$ we consider the space of tangent cone measure of μ at z_0, $T_{z_0}(\mu)$ and define $\dim(\Theta^n(\mu^0, \cdot))$ for any $\mu^0 \in T_{z_0}(\mu)$ as in §8.3. Then we have

Theorem 9.4.1 *For* $u_i \subset \mathcal{A}$*, let* $\Sigma_k = \{z_0 \in \Sigma : dim(\Theta^n(\mu^0, \cdot) \leq k, \forall \mu^0 \in T_{z_0}(\mu)\}$ *for* $0 \leq k \leq n$*. Then* $dim(\Sigma_k) \leq k$ *for* $0 \leq k \leq n$ *and* Σ_0 *is discrete.*

One can also associate a generalized $(n-2)$-varifold V_{u_i} for each u_i as in §9.1. Let V denote a limit of V_{u_i}, then all the results from §9.1 remain to be true for V. In particular,

Theorem 9.4.2 *For* L^1 *a.e.* $t \in \mathbb{R}_+$*,* $V_t \lfloor (\Sigma_t \times A_{n-2,n})$ *is a* $(n-2)$*-rectifiable varifold and* Σ_t *is a* $(n-2)$*-rectifiable set.*

For the generalized varifold flow, all the results from §9.3 remain to be true in $\mathcal{A}$. For example, we have

Theorem 9.4.3 *Under the same notations as above, if, in additions, u is a suitable weak solution to the heat equation of harmonic maps, then* $\{\nu_t\}_{t\geq 0}$ *is a Brakke flow.*

Finally, we can also prove an energy quantization for the density function of V_t as follows, whose proof is indeed slightly simpler.

Theorem 9.4.4 *If, in addition, that* $N = S^{N-1}$*, then for* $\mathcal{P}^n$ *a.e* $z_0 \in \Sigma$*,*

$$\Theta^{n-2}(\|V_{t_0}\|, x_0) = \sum_{j=1}^{l_{x_0}} E(\phi_j, S^2) \tag{9.115}$$

for some $1 \leq l_{x_0} < \infty$ *and* $\{\phi\}_{j=1}^{l_{x_0}}$ *are bubbles.*

Question 9.4.5 We believe that Theorems 9.3.1 and 9.4.4 are true for any Riemannian manifold N. This question is closely related to the problem whether the $W^{2,1}$-estimate holds for any stationary harmonic map into general target manifolds.

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